# Open-Closed String Duality for Non-critical string theory

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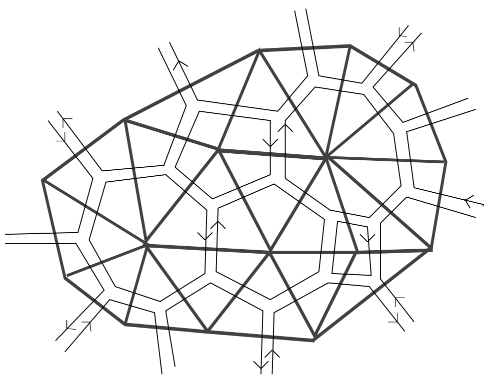
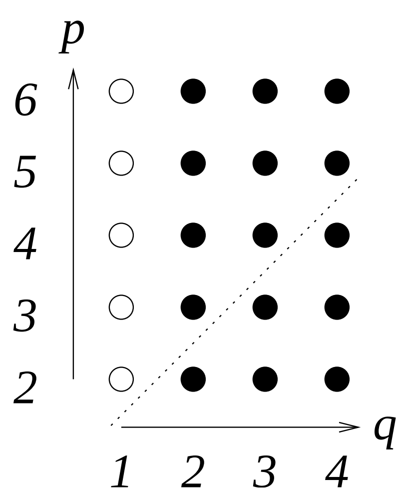
AH, Huang, Klemm, and Shih, hep-th/0501141
Ellwood and AH, hep-th/0512217

- Non-critical string theory exists for $c \leq 1$
- For minimal $(p, q)$ conformal matter and for $c = 1$,

there exists a **Double Scaled Matrix Model** formulation

$$\mathcal{Z}(t_i) = \int dA\, dB\, e^{V_p(A)+W_q(B)+AB+\sum_i t_i \mathcal{O}_i(A,B)}$$






$$c = 1 - \frac{6(p-q)^2}{pq}$$

$$\#\text{Primaries} = (p-1)(q-1)$$

These models are exactly solvable

Partition function: (e.g. for (2,1))

$$\mathcal{Z} = \tau(t_0, t_1, t_2, \ldots) = e^{\frac{t_0^3}{6} + \frac{t_1}{24} + \frac{t_0^3 t_1}{6} + \frac{1}{24} t_0 t_2 + \frac{t_1^2}{48} + \ldots}$$

- is the $\tau$-function of $p$ reduced KP heirarchy
- $Z(t_i)$ is the generating function for closed string correlation functions

**Reminiscent of Open/Closed String Duality**

# Kontsevich Matrix Model (review, Di Francesco et.al.)

e.g. for (2,1), $$\mathcal{Z}(t_i) = C(\Lambda)^{-1} \int dM e^{\mathrm{Tr}\frac{M^3}{3} + \sqrt{\Lambda} M^2}$$

$$\begin{aligned}\log \mathcal{Z} &= -\frac{1}{12}(\mathrm{Tr}\Lambda^{-1})^3 - \frac{1}{48}\mathrm{Tr}\Lambda^{-3} \\ &\quad + \frac{1}{24}(\mathrm{Tr}\Lambda^{-1})^3 \mathrm{Tr}\Lambda^{-3} + \frac{1}{32}\mathrm{Tr}\Lambda^{-1}\mathrm{Tr}\Lambda^{-5} + \frac{1}{12}(\mathrm{Tr}\Lambda^{-3})^2 + \ldots \\ &= \frac{t_0^3}{6} + \frac{t_1}{24} + \frac{t_0^3 t_1}{6} + \frac{1}{24}t_0 t_2 + \frac{t_1^2}{48} + \ldots\end{aligned}$$

$$t_{2k+1} = c_k \mathrm{Tr}\Lambda^{-(2k+1)} \qquad c_k = (-2)^{-(2k+1)/3}(2k-1)!!$$

Two distinct matrix formulations:

Both reminiscent of open-closed string duality

$$\Updownarrow$$

D-branes in minimal string theory: **ZZ, FZZT**

- Double Scaled Matrix Model $\Leftrightarrow$ ZZ-branes

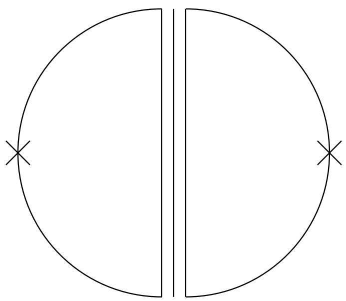
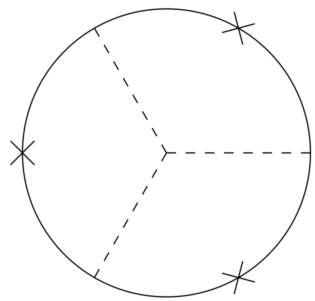
  McGreevy, Verlinde; Klebanov, Maldacena, Seiberg

- Kontsevich Matrix Model $\Leftrightarrow$ FZZT-branes

  Gaiotto, Rastelli; Maldacena, Moore, Seiberg, Shih

# **Observation of Gaiotto and Rastelli**

SFT of open strings on FZZT

$S =$  $+$  $= \int \Psi * Q\Psi + \frac{2}{3}\Psi * \Psi * \Psi$

Precisely the Kontsevich Matrix Integral

$$S = \mathrm{Tr}\left(\sqrt{\Lambda}M^2 + \frac{M^3}{3}\right), \qquad \lambda_i = \mu_{B\,i}$$

- Kontsevich Matrix Model naturally cubic
- Analogue of Witten's derivation of Chern-Simons theory as SFT of open topological A model
- The fact that the matrix model computes closed string observables is analogous to Gopakumar-Vafa
- FZZT branes are stable
- No double scaling limit

These are nice observations, but so far, only for $(p, q) = (2, 1)$ minimal models

- What about $(p, 1)$ models
- What about $c = 1$ models
- What about $\hat{c} = 1$ models
- Further generalizations?

## Generalization to $(p, 1)$

Generalized Kontsevich Matrix Model:

$$S = \mathrm{Tr}\left(\frac{Z^{p+1}}{p+1} - \Lambda Z\right)$$

Is this a SFT? Unlike (2,1), it is not cubic

- Perhaps this is an effective action after integrating out all but one open string degrees of freedom

- Perhaps this is like closed SFT or supersymmetric SFT which somehow truncates at finite order

To study the SFT along the lines of Gaiotto and Rastelli, one must investigate topological gravity with boundary, coupled to minimal topological matter Dijkgraaf, Verlinde, Verlinde

General case not as trivial as the case of pure topological gravity (2,1)

SFT/Kontsevich correspondence for $(p, 1)$ is an open problem.

## Double scaled matrix model approach

FZZT $=$ Macroscopic loop operator

$$\text{Disk} = \frac{\partial}{\partial \mu_B} \mathcal{Z}(\mu_B) = \langle \mathrm{Tr} \frac{1}{\mu_B - M} \rangle$$

$$e^{\text{Disk}} = \langle e^{\mathrm{Tr} \log(\mu_B - M)} \rangle = \langle \det(\mu_B - M) \rangle = \langle \int d\chi d\bar{\chi} e^{\bar{\chi}(\mu_B - M)\chi} \rangle$$

(2,1) model is just a Gaussian

$$\langle \ldots \rangle = \int dM (\ldots) e^{-\frac{M^2}{2g}}$$

Integrate out $M$, integrate in $(s - \bar{\chi}\chi)^2$, and integrate out $\chi$, and double scale $\Rightarrow$ Kontsevich Matrix Integral $s$

This is the observation of Maldacena, Moore, Seiberg, Shih

- Tests Kontsevich Matrix Model at the quantum level

- General lesson: Double scaled matrix model approach easier and more powerful

This approach turns out to be readily generalizable to $(p, 1)$ case

Two-Matrix Model Daul, Kazakov, Kostov

$$\mathcal{Z}(t_i) = \int dA\, dB\, e^{V_p(A)+W_q(B)+AB+\sum_i t_i \mathcal{O}_i(A,B)}$$

FZZT: $\langle \det(\lambda - B) \rangle \Rightarrow$ Generalized Kontsevich Matrix Model

That was essentially the idea of AH, Huang, Klemm, Shih

- Also issue of FZZT as a probe of target space geometry and Stokes's phenomena which de-singularizes the space time

- More importantly, it provides reasurances that SFT and Generalized Kontsevich Matrix Model should eventually come out.

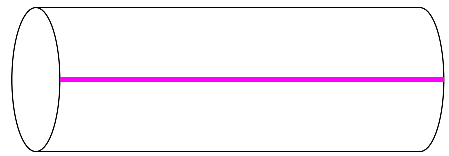
**What about the $c = 1$ model?**

Matrix model of Imbimbo and Mukhi

$$Z_n(A, \bar{t}) = (\det A)^{\nu} \int dM\, e^{\mathrm{Tr}\left(-\nu M A + (\nu - n) \log M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k\right)}$$

Is this also open/closed string duality?

• SFT for Dirichelt FZZT in $c = 1$ at self dual radius: Ghoshal, Mukhi, Murthy



$\infty$-states from winding modes: did not resemble Imbimbo-Mukhi

• Determinant operator in Normal Matrix Model: Mukherjee, Mukhi

Normal Matrix model is a matrix representation of the Toda integrable heirarchy. No double scaling. Already very close to Imbimbo-Mukhi.

Not clear how FZZT and the determinant of Normal Matrix Model are related

Lesson from before: Approach from the Double Scaled Matrix Model

$$S = \mathrm{Tr}\int_0^\beta dx \left( iP(\nabla_A \Phi) - \frac{1}{2}P^2 + \frac{1}{2}\Phi^2 \right)$$

In the Light-cone/Chiral formulation: (Kostov)

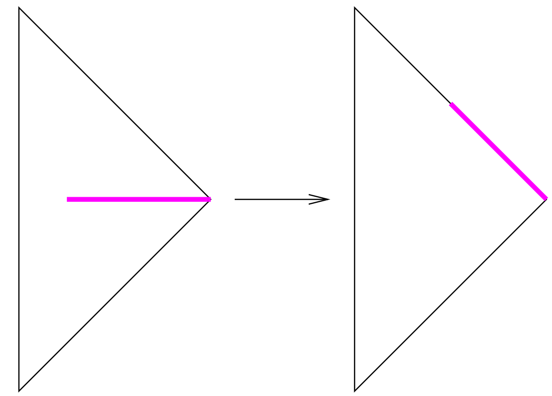
$$S = \mathrm{Tr}\int_0^\beta dx(-iX_+\nabla_A X_- + X_+X_-), \qquad X_\pm = \frac{\Phi \pm P}{\sqrt{2}}$$

FZZT-Dirichlet brane is realized by Macroscopic Loop operator

$$\begin{aligned} \det(\Phi(x) - \mu_B) &= \det\Big( (X_+(x) + X_-(x))/\sqrt{2} - \mu_B \Big) \\ &= \det\Big( (e^{ix/R}X_+(x=0) + e^{-ix/R}X_-(x=0))/\sqrt{2} - \mu_B \Big) \end{aligned}$$

Apriori depends on both $X_+$ and $X_-$. If one continues $x = -it$, and scale $\mu_B = e^t \mu'_B$, $t \to \infty$, this approaches

$$e^W \to \prod_i \det\left(1 - \frac{X_+}{\mu'_{B\,i}}\right),$$



The upshot of Ellwood and AH is that the expectation value of this operator can be massaged to take on the Imbimbo-Mukhi form.

- Explains why Ghoshal, Mukhi, Murthy didn't work. Needs scaling
- Physical meaning of scaling not 100% clear. c.f. Gaiotto, Itzhaki, Rastelli

An interesting observation: the method applicable away from self-dual radius

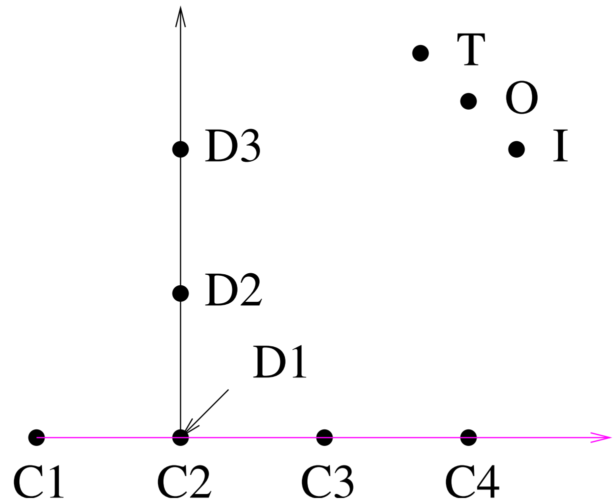
$$Z_n(A,t_-) = \int dM_i \, \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^{n} \left[ (M_i A_i)^{\nu R + (R-1)/2} e^{-\nu((M_i A_i)^R + \frac{1}{n} t_{-n} M_i^n) - n \log(M_i)} \right]$$

Integral representation of Toda flow for $R \neq 1$ (Mukherjee, Mukhi)

At $R = 1$, Harish-Chandra/Itzykson-Zuber can be used to re-write this in a matrix form

Explains why Kontsevich Matrix Model for $R \neq 1$ was not discovered

More generalizations?




Are there 1-line "matrix" representation for Toda-flows corresponding to the $D$-branch and $E$-archipelago?

How about Kontsevich-like matrix representation for the Drinfeld-Sokolov and exceptional hierarchy described in DVV?

Double scale matrix model like DKK?

How about 0A/0B models? 0A Kontsevich: (Ita, Nieder, Oz, Sakai)

$$\boxed{\int dM \, \frac{\Delta(M)}{\Delta(A)} \prod_{i=1}^{n} (M_i A_i)^{-s+R/2} J_q(\mu A_i^R M_i^R) \exp[i\mu(t_-(M_i)) - n\log(M_i)]}$$

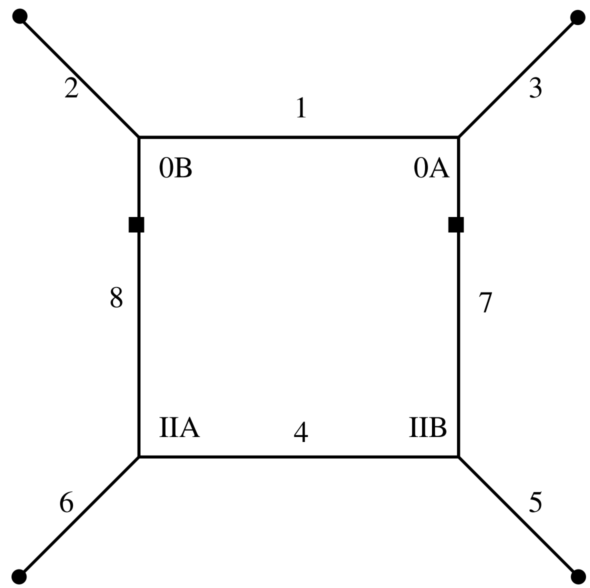
(Ellwood and AH)

0B subtle in defining the vacuum non-perturbatively

(Maldacena, Seiberg)

Everything subtle when dealing with both momentum and winding

Interesting to identify the integrable structure and the Imbimbo-Mukhi like formula for all points in the moduli-space of $\hat{c} = 1$ models (Seiberg)



How about FZZT-Neumann branes and open/closed string duality?

- Macroscopic loop operators are FZZT-Dirichlet. It sources momentum modes
- Presumably, FZZT-Neumann sources winding modes.
- A proposal due to Gaiotto:

$$S = \int dt\, \mathrm{Tr}[(D_0 M)^2 + M^2] + u^\dagger (D_0 + f(M) - \mu_B) u + \bar{u}^\dagger (D_0 + g(M) - \mu_B) \bar{u}$$

Not properly tested against disk and annulus amplitude computations (and compared against continuum)

In fact, even the Dirichlet-FZZT $\Leftrightarrow$ macroscopic loop identification is justified in part by comparing against continuum computations $\longrightarrow$ Perturbative

How does one formulate FZZT branes non-perturbatively in the double scaled matrix model?

# Parting thoughts

- How is Gaiotto-Rastelli generalized to $(p, 1)$?

- What are the generalized Kontsevich matrx integral for $D$ and $E$ type minimal/$c = 1/\hat{c} = 1$ models

- What is the physical meaning of scaling used in relating FZZT-Dirichelt branes of $c = 1$ to Imbimbo-Mukhi matrix integral?

- What is the double scaled matrix model formulation of FZZT-Neumann branes?

*Why are these issues not yet settled if the model is exactly solvable?*