A-Model Correlators from the Coulomb Branch

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Summary

- A new look at an old topological field theory: $A$-twist of $\mathcal{N} = (2, 2)$ SUSY NLSM of maps $\Phi : \mathbb{P}^1 \to X$; $X$ a smooth, compact toric variety.

  World-sheet Instantons $\leftrightarrow$ (genus 0) Gromov-Witten Invariants
  Correlators $\leftrightarrow$ Generating functions

- Witten: $X$ a \textit{phase} in the Kähler moduli space of a GLSM

- Topological correlators computed \textit{exactly} by semiclassical expansion in a particular phase;
  - Geometric Phase: instanton sums of Morrison and Plesser (sums and combinatorics)
  - Non-Geometric Phase: massive Coulomb vacua (algebraic expressions)
A GLSM is an $\mathcal{N} = (2, 2)$, $d = 2$, $G \simeq [U(1)]^r$ gauge theory;

- Parameters: F-I terms $r^a$ and $\theta$-angles $\theta^a$; $q_a := \exp(-2\pi r^a + i\theta^a)$, $a = 1, \ldots, r$.

(Bosonic) Field Content:

- $n$ Matter fields $\phi_i$, charges $Q^i_a$;
- Gauge: $v^a_i$, $\sigma^a$.

(Classical) Potential:

$$V(\phi, \sigma) = Q^a_i Q^b_i \sigma^a_i \bar{\sigma}^b_i |\phi^i|^2 + \frac{e_0}{2} D^a D^a, \quad D^a = Q^a_i |\phi^i|^2 - r^a.$$ 

$X \simeq \{ D^a = 0 \}/G$, $r^a \in K_X$ Kähler class on $X$
Twisting the GLSM and Summing the Instantons

- **GLSM** has $U(1)_+ \times U(1)_- \text{ R-symmetry};$ $U(1)_V$ non-anomalous;
- Modify energy-momentum tensor by $U(1)_V$ current; shifts spins of fermions

**Consequences:**
- Twisted theory has BRST charge $Q$, and $Q$-cohomology spanned by $\sigma_a(z)$.
- **Topological Correlators:** $\langle \sigma_{a_1}(z_1) \cdots \sigma_{a_k}(z_k) \rangle$
  - independent of world-sheet metric (RG-invariant!);
  - depend on metric of $X$ only through $q_a$.
  - Path integral localizes onto gauge instantons:
    $$\langle \sigma_{a_1} \cdots \sigma_{a_k} \rangle = \sum_n \#(n; a_1, \ldots, a_k) \prod_a q_a^{n_a}, \quad n_a = -\frac{1}{2\pi} \int f_a,$$
    $\#(n; a_1, \ldots, a_k)$: a Gromov-Witten invariant of $X$.
- Morrison and Plesser gave a toric prescription to compute $\#(n; a_1, \cdots, a_k)$ in any geometric phase.
Some Remarks on the Correlators

- Evaluation of $\#(n; a_1, \cdots, a_k)$ involves non-trivial combinatorics;
- Computing the sums can be cumbersome.

Can we circumvent these problems?

- $\langle F(\sigma) \rangle$ are rational functions of the $q_a$.
- Quantum cohomology relations:

$$\langle \prod_{i\mid Q_i^a > 0} (\xi_i)^{Q_i^a} F(\sigma) \rangle = q_a \langle \prod_{i\mid Q_i^a < 0} (\xi_i)^{-Q_i^a} F(\sigma) \rangle, \quad \xi_i = \sum_a Q_i^a \sigma_a.$$  

Is there a simple way to understand these properties?

There is a sophisticated way: Batyrev, Szenes and Vergne have shown these properties to hold by explicitly analyzing the combinatorics.

We provide a simple derivation.
Is SUSY broken in the Non-Geometric “phase”? No. Quantum vacua emerge.

Integrate out matter fields at one loop: this leads to an effective twisted superpotential for the gauge multiplets, $\tilde{W}(\Sigma)$. Equations of motion for $\sigma$:

$$\frac{\partial \tilde{W}}{\partial \sigma_a} = \prod_{i|Q_i^a>0} (\xi_i)^{Q_i^a} - q_a \prod_{i|Q_i^a<0} (\xi_i)^{-Q_i^a}.$$ 

- These vacua are reliable in the Non-Geometric region.
- Idea: in Non-Geometric region, path integral localizes onto these $\sigma$-vacua.
- We performed the localization computation in spirit of Vafa’s Topological Landau-Ginzburg Models. A $\sigma$-vacuum $\sigma_a = \hat{\sigma}_a$ contributes

$$H(\hat{\sigma}) = \left( \text{Hess } \tilde{W}(\hat{\sigma}) \prod_i \xi_i(\hat{\sigma}) \right)^{-1}$$

to the path integral.
Result and Some Applications

\[ \langle F(\sigma) \rangle = \sum_{\hat{\sigma}} F(\hat{\sigma})H(\hat{\sigma}) \]

- Rationality of correlators and quantum cohomology follow trivially,
- Often a more efficient computational procedure,
- Can use to compute A model correlators for C-Y hypersurfaces in \( X \)
- For non-compact \( X \) most phases have both gauge instantons and massive Coulomb vacua. This combination can lead to a failure of the quantum cohomology relations.
- Unlike gauge instantons, trivial to extend to genus \( g \) Riemann surface:

\[ \langle F(\sigma) \rangle_g = \sum_{\hat{\sigma}} F(\hat{\sigma})H(\hat{\sigma})^{1-g}. \]

Unfortunately, no new Gromov-Witten invariants, but a packaging for combinatorics.
Is there a pure Landau-Ginzburg description? Is it given by the Hori-Vafa Abelian duality?

Can we couple this story to topological world-sheet gravity and generate new Gromov-Witten invariants?