Recent AdS/CFT results for near-equilibrium strongly coupled thermal gauge theories

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- Viscosity-entropy ratio at non-zero chemical potential
- Critical exponents for the shear viscosity from AdS/CFT
- Thermal conductivity of N=4 SYM (Dam Son, A.S., hep-th/0601157)

- Thermal spectral functions of N=4 SYM (Pavel Kovtun, A.S., hep-th/0602059)

- Photon and dilepton production in strongly coupled plasma (Pavel Kovtun, A.S., to appear)
AdS/CFT correspondence can be used for studies of the near-equilibrium regime of strongly coupled gauge theories.

This is interesting, since this regime remains inaccessible for other non-perturbative methods such as the (direct) lattice simulations.

The Lorentzian version of the AdS/CFT computes thermal correlation functions of a dual theory directly from gravity. This is all we need since the near-equilibrium properties then follow from the fluctuation-dissipation theorems.

In particular, transport coefficients of strongly coupled thermal gauge theories can be extracted from quasinormal spectrum of the dual gravity background.
Holographically dual system in thermal equilibrium

Gravitational fluctuations

\[ g^{(0)}_{\mu\nu} + h_{\mu\nu} \]

"\( \Box \)" \( h_{\mu\nu} = 0 \) and B.C.

Quasinormal spectrum

\[ T_{\text{Hawking}} \quad S_{\text{Bekenstein-Hawking}} \quad \leftrightarrow \quad T \quad S \]

Deviations from equilibrium

\[ j_i = -D \partial_i j^0 + \cdots \]

\[ \partial_t j^0 + \partial_i j^i = 0 \]

\[ \partial_t j^0 = -D \nabla^2 j^0 \]

\[ \omega = -iDq^2 + \cdots \]
Transport (kinetic) coefficients

- Shear viscosity $\eta$
- Bulk viscosity $\varsigma$
- Charge diffusion constant $D_Q$
- Thermal conductivity $\kappa_T$
- Electrical conductivity $\sigma$
What is known?

✓ Shear viscosity/entropy ratio: \[ \frac{\eta}{s} = \frac{1}{4\pi} \]
  • in the limit \( g^2N = \infty \quad N = \infty \)
  • universally for a large class of theories

✓ Bulk viscosity for non-conformal theories
  \[ \frac{\zeta}{\eta} = -\kappa \left( v_s^2 - \frac{1}{3} \right) \]
  • in the limit \( g^2N = \infty \quad N = \infty \)
  • model-dependent

✓ R-charge diffusion constant for N=4 SYM: \[ D_R = \frac{1}{2\pi T} \]
Shear viscosity in $\mathcal{N} = 4$ SYM

$\sim \frac{1}{\lambda^2 \log \frac{1}{\lambda}}$

(perturbative thermal gauge theory)

$\frac{1}{4\pi} + \frac{135\zeta(3)}{32\pi} \frac{1}{(2\lambda)^{3/2}} + \cdots$

Correction to $1/4\pi$: A.Buchel, J.Liu, A.S., hep-th/0406264
Shear viscosity at non-zero chemical potential

\[ \mathcal{N} = 4 \text{ SYM} \]

\[ q_i \in U(1)^3 \subset SO(6)_R \]

\[ Z = \text{tr} e^{-\beta H + \mu_i q_i} \]

We still have

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]

\[ \eta = \pi N^2 T^3 \frac{m^2(1 - \sqrt{1 - 4m^2 - m^2})^2}{(1 - \sqrt{1 - 4m^2})^3} \]

\[ m \equiv \frac{\mu}{2\pi T} \]

Reissner-Nordstrom-AdS black hole with three R charges

(Behrnd, Cvetic, Sabra, 1998)

J.Mas
D.Son, A.S.
O.Saremi
K.Maeda, M.Natsuume, T.Okamura
Thermal conductivity

Non-relativistic theory: \[ Q = -\kappa_T \nabla T \]

Relativistic theory: \[ T^{0i} = -\kappa_T \left( \partial^i T - \frac{T}{\varepsilon + P} \partial^i P \right) \]

Kubo formula: \[ \kappa_T = -\frac{(\varepsilon + P)^2}{\rho^2 T} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G(\omega, 0) \]

In $\mathcal{N} = 4$ SYM with non-zero chemical potential $\mu$:

\[ \frac{\kappa_T \mu^2}{\eta T} = 8\pi^2 \]

One can compare this with the Wiedemann-Franz law for the ratio of thermal to electric conductivity:

\[ \frac{\kappa_T e^2}{\sigma T} = \frac{\pi^2}{3} \]
Spectral function and quasiparticles

\[ \chi_{\mu\nu,\alpha\beta}(k) = \int d^4x \ e^{-ik\cdot x} \left[ T_{\mu\nu}(x) T_{\alpha\beta}(0) \right] \]

A: scalar channel
B: scalar channel - thermal part
C: sound channel
Lattice test of the viscosity/entropy bound $\eta/s \geq 1/4\pi$?

A. Nakamura, S. Sakai, hep-lat/0510039
Photon and dilepton emission from strongly coupled YM plasma
Computing the emission rate

1. In N=4 SYM, gauge $U(1)_R \subset SU(4)_R$ with $\alpha_{em} \ll 1$

2. Cancel the anomaly by adding weakly interacting, non-thermal fermions

3. The emission rate is
   \[
   \frac{d\Gamma}{d^4x d^3q} = \frac{\alpha_{em} \eta^{\mu\nu}}{(2\pi)^2} \Pi_{\mu\nu}(\omega, q)
   \]

   \[
   \Pi_{\mu\nu}(\omega, q) = \int d^4x e^{-i\omega t + iq \cdot x} \langle J_\mu(0) J_\nu(x) \rangle_T
   \]

4. The Wightman correlator is computed from gravity
Photoproduction rate

\[ \frac{d^2 \Gamma}{dk^2} / \alpha_s \alpha_{\text{EM}} T^3 \]

Photon production rate

- Total
- Bremsstrahlung
- Pair
- $2 \leftrightarrow 2$

Perturbative QCD

(Arnold, Moore, Yaffe, 2001)

N=4 SYM ++
Outlook

- How universal is $\eta/s$?
- How useful are the N=4 spectral functions for thermal QCD lattice simulations?
- Can we get a meaningful comparison of photon and lepton production rates obtained using pQCD, lattice, AdS/CFT, RHIC?
The hydrodynamic regime

Hierarchy of times (example)

Mechanical description
Kinetic theory
Hydrodynamic approximation
Equilibrium thermodynamics

Hierarchy of scales

$l_{mfp} \ll l \ll L$

(L is a macroscopic size of a system)
What is viscosity?

Friction in Newton’s equation:
\[
\frac{d(mv_i)}{dt} + \gamma v_i = F_i
\]

Friction in Euler’s equations
\[
\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial}{\partial x^k} (P\delta_{ik} + \rho v_i v_k) + \frac{\partial}{\partial x^k} \sigma_{ik}^{\text{fric}}
\]

\[
\sigma_{ik}^{\text{fric}} \sim \frac{\partial v_i}{\partial x^k} \\
\sigma_{ik}^{\text{fric}} \sim \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i}
\]

\[
\sigma_{ik}^{\text{fric}} = \eta \left( \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \frac{2}{d} \delta_{ik} \frac{\partial v_l}{\partial x^l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x^l} + \ldots
\]
Viscosity of gases and liquids

Gases (Maxwell, 1867):

\[ \eta \sim \rho \bar{v} l_{mf} \sim \frac{m_0 \bar{v}}{\sigma} \sim \frac{m_0^{1/2}}{\sigma} \sqrt{T} \]

Viscosity of a gas is

- independent of pressure
- scales as square of temperature
- inversely proportional to cross-section

Liquids (Frenkel, 1926):

\[ \eta \sim A(P, T) \exp \frac{W}{T} \]

- \( W \) is the “activation energy”
- In practice, \( A \) and \( W \) are chosen to fit data
Computing transport coefficients from “first principles”

Fluctuation-dissipation theory
(Callen, Welton, Green, Kubo)

Kubo formulae allows one to calculate transport coefficients from microscopic models

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, d^3 x e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

In the regime described by a gravity dual the correlator can be computed using AdS/CFT
Universality of shear viscosity in the regime described by gravity duals

\[ ds^2 = f(\omega) \left( dx^2 + dy^2 \right) + g_{\mu \nu}(\omega) dw^\mu dw^\nu \]

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, dx e^{i\omega t} \left\langle \left[ T_{xy}(t, x), T_{xy}(0, 0) \right] \right\rangle \]

\[ \sigma_{abs} = -\frac{16\pi G}{\omega} \, \text{Im} \, G^R(\omega) \]

\[ = \frac{8\pi G}{\omega} \int dt \, dx e^{i\omega t} \left\langle \left[ T_{xy}(t, x), T_{xy}(0, 0) \right] \right\rangle \]

Graviton’s component \( h_{xy}^x \) obeys equation for a minimally coupled massless scalar. But then \( \sigma_{abs}(0) = A_H \).

Since the entropy (density) is \( s = A_H / 4G \) we get

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]
Three roads to universality of $\eta/s$

- **The absorption argument**
  D. Son, P. Kovtun, A.S., hep-th/0405231

- **Direct computation of the correlator in Kubo formula from AdS/CFT**
  A.Buchel, hep-th/0408095

- **“Membrane paradigm” general formula for diffusion coefficient + interpretation as lowest quasinormal frequency = pole of the shear mode correlator + Buchel-Liu theorem**
  P. Kovtun, D.Son, A.S., hep-th/0309213, A.S., to appear,
A viscosity bound conjecture

\[
\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \approx 6.08 \cdot 10^{-13} \, K \cdot s
\]
Hydrodynamics as an effective theory

Thermodynamic equilibrium:  \[ \langle T^{00} \rangle = \epsilon, \quad \langle T^{0i} \rangle = 0 \]
\[ T^{ij} = P(\epsilon) \delta^{ij} \]

Near-equilibrium:
\[ T^{00} = \epsilon + \tilde{T}^{00} \]
\[ T^{ij} = P \delta^{ij} + \left( \frac{\partial P}{\partial \epsilon} \right) \tilde{T}^{00} + \tilde{T}^{ij} \]
\[ \tilde{T}^{ij} = -\frac{1}{\epsilon + P} \left[ \eta \left( \partial_i \tilde{T}^{0j} + \partial_j \tilde{T}^{0i} - \frac{2}{3} \delta^{ij} \partial_k \tilde{T}^{0k} \right) + \zeta \delta^{ij} \partial_k \tilde{T}^{0k} \right] + \ldots \]

Eigenmodes of the system of equations
\[ \partial_\mu T^{\mu\nu} = 0 \]

Shear mode (transverse fluctuations of \( \tilde{T}^{0i} \)):
\[ \omega = -\frac{i \eta}{\epsilon + P} q^2 \]

Sound mode:
\[ \omega = v_s q - \frac{i}{2 \epsilon + P} \left( \zeta + \frac{4}{3} \eta \right) q^2 \]

For CFT we have \( \zeta = 0 \) and \( \epsilon = 3P \) \[ v_s = \frac{1}{\sqrt{3}} \]
Two-point correlation function of stress-energy tensor

Field theory

Zero temperature:
\[ \langle T_{\mu\nu}T_{\alpha\beta}\rangle_{T=0} = \Pi_{\mu\nu,\alpha\beta} F(k^2) + Q_{\mu\nu,\alpha\beta} G(k^2) \]

Finite temperature:
\[ \langle T_{\mu\nu}T_{\alpha\beta}\rangle_T = S_{\mu\nu,\alpha\beta}^{(1)} G_1(\omega, q) + S_{\mu\nu,\alpha\beta}^{(2)} G_2(\omega, q) \]
\[ + S_{\mu\nu,\alpha\beta}^{(3)} G_3(\omega, q) + S_{\mu\nu,\alpha\beta}^{(4)} G_4 + S_{\mu\nu,\alpha\beta}^{(5)} G_5 \]

Dual gravity

- Five gauge-invariant combinations \( Z_1, Z_2, Z_3, Z_4, Z_5 \) of \( h_{\mu\nu} \) and other fields determine \( G_1, G_2, G_3, G_4, G_5 \)
- \( Z_1, Z_2, Z_3, Z_4, Z_5 \) obey a system of coupled ODEs
- Their (quasinormal) spectrum determines singularities of the correlator
Classification of fluctuations and universality

\[ ds^2 = \frac{r^2}{R^2} \left( -f(r)dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{R^2}{r^2} f dR^2 \]

\[ \delta g_{\mu \nu} \sim e^{-i\omega t + iqz} h_{\mu \nu}(r) \]

O(2) symmetry in x-y plane

Shear channel: \( h_{tx} \quad h_{zx} \quad h_{ty} \quad h_{zy} \)

Sound channel: \( h_{tt} \quad h_{tz} \quad h_{zz} \quad h_{xx} + h_{yy} \)

Scalar channel: \( h_{xy} \quad h_{xx} - h_{yy} \)

Other fluctuations (e.g. \( \delta \varphi_1, \ldots \delta \varphi_n \)) may affect sound channel

But not the shear channel \( \quad \) universality of \( \eta/s \)
Gauge-invariant variables for a gravity dual to a conformal theory

\[ ds^2 = a(r) \left( -f(r)dt^2 + dx^2 + dy^2 + dz^2 \right) + b(r)dr^2 \]

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \]

Shear: \[ Z_1 = q H_{tx} + \omega H_{zx} \]

Sound: \[ Z_2 = q^2 f H_{tt} + 2\omega q H_{tz} + \omega^2 H_{zz} + q^2 f \left( 1 + \frac{af'}{a'f} - \frac{\omega^2}{q^2 f} \right) H \]

Scalar: \[ Z_3 = H_{xy} \]

\[ H_{ij} = h_{ij}/a \quad h_{\mu\nu} \sim e^{-i\omega t + iqz} \quad H = (h_{xx} + h_{yy}) / 2a \]
Bulk viscosity and the speed of sound in $\mathcal{N} = 2^*$ SYM

$\mathcal{N} = 2^*$ is a “mass-deformed” $\mathcal{N} = 4$ (Pilch-Warner flow)

- Finite-temperature version: A.Buchel, J.Liu, hep-th/0305064
- The metric is known explicitly for $m/T \ll 1$
- Speed of sound and bulk viscosity:

$$v_s = \frac{1}{\sqrt{3}} \left( 1 - \frac{[\Gamma (\frac{3}{4})]^4}{3\pi^4} \left( \frac{m_f}{T} \right)^2 - \frac{1}{18\pi^4} \left( \frac{m_b}{T} \right)^4 + \cdots \right)$$

$$\frac{\zeta}{\eta} = \beta_f \frac{[\Gamma (\frac{3}{4})]^4}{3\pi^3} \left( \frac{m_f}{T} \right)^2 + \frac{\beta_b^r}{432\pi^2} \left( \frac{m_b}{T} \right)^4 + \cdots$$

$$\frac{\zeta}{\eta} = -\kappa \left( v_s^2 - \frac{1}{3} \right)$$
Heavy ion collisions: RHIC/LHC
QCD phase diagram
QCD deconfinement transition

\[ \frac{\varepsilon}{T^4} \]

$T_c \approx (173 \pm 15) \text{ MeV}$

\[ \varepsilon_c \approx 0.7 \text{ GeV/fm}^3 \]

SPS, RHIC, LHC
Pressure in perturbative QCD

![Graph showing the relationship between pressure and temperature in perturbative QCD.](image)
Figure from: U. Heinz, “Concepts of heavy-ion physics”, hep-ph/0407360
Elliptic flow at RHIC

![Graph showing elliptic flow as a function of p_T (GeV) for different experiments: STAR, PHENIX, EOS Q, and EOS H. The graph plots V_2 (%) on the y-axis against p_T (GeV) on the x-axis. The data points and curves indicate the observed elliptic flow patterns.]
Effect of viscosity on elliptic flow
Conclusions

- AdS/CFT gives insights into physics of thermal gauge theories in the nonperturbative regime.
- Generic hydrodynamic predictions can be used to check validity of AdS/CFT.
- General algorithm exists to compute transport coefficients and the speed of sound in any gravity dual.
- Model-independent statements can presumably be checked experimentally.