# Semiclassical strings in $AdS_5 \times S^5$ and AdS/CFT

some related papers:

J. A. Minahan, A. Tirziu and A. A. Tseytlin, “1/J corrections to semiclassical AdS/CFT states from quantum Landau-Lifshitz model,” hep-th/0509071;

“$1/J^2$ corrections to BMN energies from the quantum long range Landau-Lifshitz model,” hep-th/0510080.

N. Beisert and A. A. Tseytlin, “On quantum corrections to spinning strings and Bethe equations,” hep-th/0509084

# AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$ dual to

free type IIB superstrings in $AdS_5 \times S^5$

Parameters:

$\lambda = g^2_{YM} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

One implication of duality:

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

$J$ - global charges of $SO(2,4) \times SO(6)$:

spins $S_1, S_2;\ J_1, J_2, J_3$

$m$ - extra quantum numbers like winding numbers, number of folds, oscillation numbers, ...

Operators: Tr$(\prod_i \Phi_i^{J_i} D_+^{S_1} D_*^{S_2} ... F_{mn} ... \Psi ...)$

Solve SYM/string theory $\to$

compute $E = \Delta$ for any $\lambda$ (and $J$,$m$)

Solve non-trivial max. susy 4-d CFT = string in curved R-R background

Remarkable well-defined problem of mathematical physics

hope to learn more about less susy theories, e.g., role of integrability and string picture in perturbative / large energy QCD

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

“Constructive” approach:

use perturbative results on both sides and other properties (integrability,...) as guides to exact answers (Bethe ansatze,...)

Recent progress:

“semiclassical” states with large quantum numbers ($J \gg 1$)

dual to “long” gauge operators:

same dependence on $J$ with coefficients = interpolating functions of $\lambda$

connection to spectrum of integrable spin chains

– anomalous dimensions on gauge side

Semiclassical strings in $AdS$: oscillating, rotating, etc:

$E = N + ...$, or $E = S + ...$ (de Vega, Sanchez, 1994)

while $E \sim \sqrt{N}$ in flat space

Generic long quantum rotating/pulsating string in $AdS_5 \times S^5$

$$E = f(\lambda) J + ..., \qquad J \gg 1$$

dual to long operators like Tr$(\bar{\Phi}\Phi...)$

or

$$E = S + f(\lambda) \ln S + ... , \qquad S \gg 1$$

dual to Tr$(\Phi D^S \Phi)$

in both cases:

$$f(\lambda \gg 1) = a_1 \sqrt{\lambda} + a_2 + \frac{a_3}{\sqrt{\lambda}} + ...$$

$$f(\lambda \ll 1) = k_1 \lambda + k_2 \lambda^2 + ...$$

Generic pattern: non-trivial "interpolating" functions of $\lambda$

Other examples:

"$q-\bar{q}$" potential (conformal theory !):

$$V(r)=\frac{f(\lambda)}{r}$$

$f(\lambda\gg 1)=c_1\sqrt{\lambda}+c_2+\dots\,,\qquad c_1=-\frac{4\pi^2}{[\Gamma(1/4)]^4}$

SYM entropy:

$$S=\frac{2\pi^2}{3}f(\lambda)N^2T^3V_3$$

$f(\lambda\ll 1)=1-\frac{3}{2\pi^2}\lambda+\frac{3+\sqrt{2}}{\pi^2}\sqrt{\lambda}+\dots$

$f(\lambda\gg 1)=\frac{3}{4}+\frac{45}{32}\zeta(3)\frac{1}{(\sqrt{\lambda})^3}+\dots$

General properties of $f(\lambda)$?

Sign-alternating series with finite radius of convergence? ('t Hooft, large $N$)

Indicative example:

folded string rotating at the center of $AdS$ (de Vega, Egusquiza, 1996)

$$\Delta=S+f(\lambda)\ln S+\dots$$

$$f(\lambda \gg 1) = a_1\sqrt{\lambda} + a_2 + \frac{a_3}{\sqrt{\lambda}} + ...$$

dual to Tr$\left(F_{+m}D_+^{S-4}F_{+m}\right) + ...,\quad S \gg 1$

$a_1 = \frac{1}{\pi}$ (Gubser, Klebanov, Polyakov (2002))

$a_2 = -\frac{3}{\pi}\ln 2$ (Frolov, A.T. (2002))

$$f(\lambda \ll 1) = k_1\lambda + k_2\lambda^2 + ...$$

$k_1 = \frac{1}{2\pi^2},\quad k_2 = -\frac{1}{96\pi^2},\quad k_3 = \frac{11}{23040\pi^2}, ...$

(perturbative QCD/SYM)

Approximate Pade interpolation (Lipatov et al, 2004):

$f(\lambda) \approx \frac{12}{\pi^2}(\sqrt{1+\frac{\lambda}{12}} - 1)$

Branch cut from $(-\infty, -12)$

more likely hypergeometric function e.g. ${}_2F_1(a, b, c, -\frac{\lambda}{\pi^2})$, ...

## Semiclassical strings in $AdS_5$

spiky strings (Kruczenski, 2004):

$$\Delta = S + mf(\lambda)\ln\frac{S}{m} + ...$$

dual to Tr$\left(D_+^{s_1}FD_+^{s_2}F...D_+^{s_m}F\right),\quad S = s_1 + s_2 + ... + s_m$

Smooth strings without folds/spikes:

dual to operators without covariant derivatives
Tr$(F_{n_1 k_1} F_{n_2 k_2} ... F_{n_L k_L})$

1-loop closed sector in YM if $F_{nk}$ is selfdual

1-loop dilatation operator: Hamiltonian of

antiferromagnetic $\text{XXX}_1$ ($s = -1, 0, 1$) spin chain (Heise, Ferreti, Zarembo, 2005)

$SO(4)$: $(S_1, S_2)$, $\quad S_L = S_1 + S_2, \;\; S_R = S_1 - S_2$

Antiferromagnetic state: $S_L = 0$

Ferromagnetic state: $S_L = S, \quad S_R = 0$

Anomalous dimension:

$$\Delta = f(\lambda) S = S + c_1 \lambda S + ..., \qquad S \gg 1$$

Dual to which strings?

Circular string rotating in two orthogonal planes:

$S_1 = S_2 = S/2$ (Frolov, A.T., 2003)

$$E = S + (\lambda S)^{1/3} + ...$$

1-loop string correction – linear in $S$ in the region of stability

(Park, Tirziu, A.T., 2005)

$$E = a_0 S + \frac{1}{\sqrt{\lambda}} a_1 S + ..., \qquad S \gg 1$$

Supports identification with ferromagnetic state on gauge side

Reducing spin– pulsations in radial direction of AdS:

antiferromagnetic (spin 0) state corresponds to string pulsating in $S^3$ part of $AdS_5$

## Special classes of semiclassical string states in $S^5$

strings moving in $S^5$ – operators built of SYM scalars $\Phi_i$

BPS: pointlike string along big circle of $S^5$

dual to Tr$\Phi^J$: $E = \Delta = J$ (protected by susy)

near-BPS: small strings with large c.o.m. momentum $J$

dual to near-BPS (Berenstein-Maldacena-Nastase) operators Tr $(\Phi^J ...)$

energies/dimensions match to leading order in $1/J$

to all orders in $\tilde{\lambda} = \frac{\lambda}{J^2}$

large strings which are "locally BPS":

"Fast" strings = multi-spin strings fast-rotating in $S^5$

nearly-null world surface (cf. ray of BPS geodesics – as if tension=0)

$$E_{class} = \sqrt{\lambda}\mathcal{E}(\mathcal{J}) = J + a_0\frac{\lambda}{J} + b_0\frac{\lambda^2}{J^3} + ...$$
$$= J(1 + a_0\tilde{\lambda} + b_0\tilde{\lambda}^2 + ...)$$

$\lambda \gg 1, \quad J \equiv \sqrt{\lambda}\mathcal{J} \gg 1,$

$\tilde{\lambda} \equiv \frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$=fixed

Dual to “locally-BPS” long operators, e.g.

Tr$(\Phi_1^{J_1}\Phi_2^{J_2} + ...)$ $\approx$Tr$(n_a\Phi_a)^J$

$\approx$ special low-energy states of spin chain for dilatation operator

Limits:

## BMN limit:

$J_1 \gg 1, \ J_2$=finite:

“short” strings with c.o.m. along $S^5$ geodesic

dual to Tr$([\Phi_1....\Phi_1]\Phi_2[\Phi_1....\Phi_1]\Phi_2...)$

with $J_2$ “impurities”

## “Thermodynamic” or “multi-spin” limit:

$J_1 \gg 1 \ , \quad J_2 \gg 1 \ , \ \frac{J_1}{J_2}$=finite

(Frolov, A.T., 2003)

Long fast strings dual to long “spin-wave” operators

Tr$([\Phi_1....\Phi_1][\Phi_2...\Phi_2][\Phi_1....\Phi_1][\Phi_2...\Phi_2]...)$

planar 1-loop dilatation operator of $\mathcal{N} = 4$ SYM

= Hamiltonian of ferromagnetic Heisenberg $\text{XXX}_{1/2}$ spin chain

$$H_1 = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1})$$

$E = J + \lambda E_1 + ..$

Spectrum (long length $J$):

$E_1 = 0$: ferromagnetic vacuum – BPS operator (point-like string)

$E_1 \sim \frac{1}{J^2}$: magnons – BMN states (short strings)

$E_1 \sim \frac{1}{J}$: low-energy spin waves (fast long strings)

$E_1 \sim J$: anti-ferromagnetic state and near-by spinons (“slow” strings)

Higher orders (Beisert,Kristjansen,Staudacher, 2003; Beisert, 2004):

$$H_2 = \frac{\lambda^2}{(4\pi)^4} \sum_{l=1}^{J} (-3 + 4\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2})$$

$H_3$ contains $\vec{\sigma}_l \cdot \vec{\sigma}_{l+3}$ and also $(\vec{\sigma}_l \cdot \vec{\sigma}_{l+1})(\vec{\sigma}_{l+2} \cdot \vec{\sigma}_{l+3})$, etc.

“long-range” ferromagnetic spin chain

[ H= effective Hamiltonian for Hubbard model:

Rej, Serban, Staudacher, 2005 ]

## Known structure of $E$ and $\Delta$

String side: classical + quantum $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections

first large $\lambda$ or large $J$ at fixed $\tilde{\lambda}$, then expanded in $\tilde{\lambda}$

$$E = J\Big[1 + \tilde{\lambda}(a_0 + \frac{a_1}{J} + \frac{a_2}{J^2} + ...) + \tilde{\lambda}^2(b_0 + \frac{b_1}{J} + \frac{b_2}{J^2} + ...)$$

$$+\tilde{\lambda}^3(c_0 + ...) + ...\Big]$$

perturbative SYM: first small $\lambda$, then expand in large $J$

$$\Delta = J + \lambda(\frac{d_1}{J} + \frac{d_2}{J^2} + ...) + \lambda^2(\frac{e_1}{J^3} + \frac{e_2}{J^4} + ...) + \lambda^3(\frac{h_1}{J^5} + ...) + ...$$

two expansions seem to have same structure!

apparent existence of thermodynamic scaling limit on gauge side –

non-trivial consequence of susy

Moreover, leading coefficients happen to match precisely

(Frolov, A.T., 2003; Beisert,Minahan,Staudacher,Zarembo, 2003)

$a_0 = d_1, \quad b_0 = e_1, \quad a_1 = d_2, \quad b_1 = e_2$

e.g., 1-loop string coeff. = $1/J$ (finite size) correction on spin chain side
(Beisert, A.T., Zarembo, 2005)

But $c_0 \neq h_1$ ?!

general pattern should not be exact matching but rather weak to strong coupling interpolation

Indeed, the limits are different

string side :

first $\lambda \gg 1$ at fixed $\tilde{\lambda} = \frac{\lambda}{J^2}$, then expand in $\tilde{\lambda}$

gauge side :

first $\lambda \ll 1$ at fixed $J$, then expand in $1/J$

## Resolution of "3-loop discrepancy":

Quantum string expansion near fast or BMN strings

contains "non-analytic" terms with explicit factors of $\sqrt{\lambda}$

(Beisert, A.T., 2005)

matching to perturbative gauge theory expansion beyond $\lambda^2$ order

only in a proper limit

$$E = J\left[1 + \tilde{\lambda}(a_0 + ...) + \tilde{\lambda}^2(b_0 + ...) + \tilde{\lambda}^3(f(\lambda) + ...) + ...\right]$$

interpolating function

$$f(\lambda) = c_0 + \frac{c_1}{\sqrt{\lambda}} + \dots ,$$

$f(\lambda \gg 1) = c_0 \neq f(\lambda \ll 1) = h_1$

Similarly for "M-impurity" BMN states:

was expected

$$E(M) = J - M + \sum_{j=1}^{M} \sqrt{1 + \tilde{\lambda} n_j^2} + \frac{F_1(\tilde{\lambda}, n_i)}{J} + \frac{F_2(\tilde{\lambda}, n_i)}{J^2} + \dots$$

with $F_i$ polynomial in $\tilde{\lambda}$

but in fact they contain $\sqrt{\tilde{\lambda}}$ or

$$F_1(\tilde{\lambda}, n_i) = \tilde{\lambda} q_1(n_i) + \tilde{\lambda}^2 q_2(n_i) + \tilde{\lambda}^3 f(\lambda) q_3(n_i) + \dots$$

$f(\lambda \gg 1) \neq f(\lambda \ll 1)$

(Beisert, A.T.; Minahan, Tirziu, A.T., 2005)

How to compute "interpolating functions"?

Structure of underlying Bethe ansatz?

Superstring world-sheet S-matrix?

## Effective field theory approach:

How to explain/understand leading-order matching between string and gauge energies, states, higher conserved charges ?

two microscopically consistent theories

– spin chain and superstring –

approximated by low-energy 2d effective actions

lead to non-relativistic "Landau-Lifshitz" 2d action

describing spin chain and string world-sheet d.o.f.

works not only classically but also at the quantum level

supports of integrability of $AdS_5 \times S^5$ superstring at the quantum level

reveals "spin chain" structure of quantum string

$\lambda \gg 1$ to $\lambda \ll 1$ interpolation between "string" and "gauge" effective actions and corresponding "spin chains"

Coherent-state action for low-energy excitations of spin chain (determined by H = dilatation operator)

and "fast-string" limit of string action

$\vec{n}$ – transverse position of string or spin coherent state $(\vec{n}^2 = 1)$

Continuum limit:

Classical Landau-Lifshitz equations of motion

$$\partial_t n_i = \frac{1}{2}\tilde{\lambda}\epsilon_{ijk} n_j \partial_\sigma^2 n_k$$

describing low-energy part of the spectrum with energies

$\sim J\tilde{\lambda} = \frac{\lambda}{J}$

(lowest order in quantum $1/J$ expansion)

Integrable system: Lax pair, finite gap solution, etc.

# Beyond $\tilde{\lambda}^2$ order: weak-strong coupling interpolation

effective actions from gauge-theory spin chain and string theory:

$$S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi}\, L$$

$$L = \vec{C}(n)\cdot\partial_t\vec{n} - \frac{1}{4}\vec{n}\left(\sqrt{1-\tilde{\lambda}\partial_\sigma^2}-1\right)\vec{n} - \frac{3}{128}\tilde{\lambda}^2(\partial_\sigma\vec{n})^4$$

$$-\frac{\tilde{\lambda}^3}{64}\Big[-\frac{7}{4}(\partial_\sigma\vec{n})^2(\partial_\sigma^2\vec{n})^2 + \mathrm{b}(\lambda)\,(\partial_\sigma\vec{n}\partial_\sigma^2\vec{n})^2 + \mathrm{c}(\lambda)\,(\partial_\sigma\vec{n})^6\Big]+...$$

quadratic part is exact: reproduces the BMN dispersion relation for small ("magnon") fluctuations near the BPS vacuum $\vec{n}=(0,0,1)$.

Orders $\tilde{\lambda}$ and $\tilde{\lambda}^2$:

agreement between two effective actions $\rightarrow$ explains observed agreement of energies of states, integrable structures, etc.

finite-size $1/J$ corrections on spin chain side: non-trivial

quantum $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections on superstring side: hard

Quantizing LL action provides a short-cut to exact results

Effective field theory philosophy: underlying microscopic UV finite theories (spin chain and string ) dictate particular choice of regularization of quantum LL theory

extra modes (bosons outside $S^3$ LL sector and fermions) are “heavy” – can be integrated out – play the role of UV regulators

Both spin chain and string theory lead to the same $1/J$ quantum-corrected effective field theory at orders $\lambda$ and $\lambda^2$

(Minahan, Tirziu, A.T, 2005)

“3-loop” coefficients in the string and gauge theory expressions

$$\mathrm{b}_s = -\frac{25}{2} , \qquad \mathrm{c}_s = \frac{13}{16} ,$$

$$\mathrm{b}_g = -\frac{23}{2} , \qquad \mathrm{c}_g = \frac{12}{16}$$

Quantum string effective action: include $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections:

$$\mathrm{b}(\lambda) = \mathrm{b}_s + \frac{p_1}{\sqrt{\lambda}} + ..., \qquad \mathrm{c}(\lambda) = \mathrm{c}_s + \frac{q_1}{\sqrt{\lambda}} + ..., \qquad \lambda \gg 1 ,$$

for small $\lambda$ should approach gauge values

agrees with the structure of non-analytic terms in 1-loop string correction:

$$J \times \tilde{\lambda}^3 \times \frac{1}{\sqrt{\lambda}} = \frac{\lambda^{5/2}}{J^5}$$

# All-order gauge theory Bethe ansatz

(Beisert,Dippel,Staudacher, 2004)

asymptotic: up to $\lambda^J$

$$e^{ip_i J} = \prod_{j\neq i}^{M} \frac{u_i - u_j + i}{u_i - u_j - i}\,,$$

$$u_i = \frac{1}{2}\cot\frac{p_i}{2}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_i}{2}}$$

$$E = \sum_{i=1}^{M}\Big(\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_i}{2}} - 1\Big)$$

$p_i$ = momenta of magnons that may form bound states

To match quantum string need extra S-matrix phase factor:

$e^{i\phi(\lambda,p_i)}$ containing interpolating functions of coupling $\lambda$

with $\phi(\lambda\to 0, p_i)\to 0$

“string Bethe ansatz” (Arutyunov, Frolov, Staudacher, 2004)

# Antiferromagnetic end of the spectrum

all spin-chain states should have string counterparts

dimensions of gauge operators bound from above

by antiferromagnetic ($J_1 = J_2$) state of XXX chain

Similar bound should appear in quantum string theory

corresponding string state? Can be seen in semiclassical approximation? [work in progress with Tirziu]

Hint from strong-coupling extrapolation of AF state of BDS chain:

anom. dim. of AF state operator Tr$\left(\Phi_1^{J/2}\Phi_2^{J/2}\right)$ (Zarembo, 2005)

$$\Delta = f(\lambda)\, J$$

$$f(\lambda) = 1 + \frac{\sqrt{\lambda}}{\pi} \int_0^\infty \frac{dk}{k} \, \frac{\mathbf{J}_0\left(\frac{\sqrt{\lambda}\,k}{2\pi}\right)\mathbf{J}_1\left(\frac{\sqrt{\lambda}\,k}{2\pi}\right)}{\mathrm{e}^{\,k} + 1}$$

small $\lambda$:

$$f(\lambda \ll 1) = 1 + 4\ln 2\, \frac{\lambda}{16\pi^2} - 9\zeta(3)\left(\frac{\lambda}{16\pi^2}\right)^2 + 75\zeta(5)\left(\frac{\lambda}{16\pi^2}\right)^3 + \ldots$$

large $\lambda$ extrapolation:

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi^2} + \ldots$$

suggests that $E \sim \sqrt{\lambda} J$ for dual string state

Simplest circular rotating string 2-spin solution (Frolov, A.T., 2003)

on $S^3$ part of $S^5$: $|\mathrm{X}_1|^2 + |\mathrm{X}_2|^2 = 1$

$$\mathrm{X}_1 = a e^{i(w\tau + m\sigma)}, \qquad \mathrm{X}_1 = a e^{i(w\tau - m\sigma)}, \qquad a = \frac{1}{\sqrt{2}}$$

$J = \sqrt{\lambda}\, w$ and $J_1 = J_2 = J/2$ , $m$=integer winding

$$E = \sqrt{\lambda}\sqrt{w^2 + m^2}, \quad i.e. \quad E = \sqrt{J^2 + \lambda m^2}$$

string in flat space: $m = w$, $a$=arbitrary

string on a sphere: can be static – $w = 0$, wrapped on circle (unstable)

Semiclassical expansion:

$\lambda \gg 1$, $w, m$=fixed

Fast string limit: $w \gg m$: $\tilde{\lambda} = \frac{\lambda}{J^2} \ll 1$

$$E = J + \frac{\lambda m^2}{2J} + O(\lambda^2) = J\big[1 + \frac{1}{2}\tilde{\lambda} m^2 + O(\tilde{\lambda}^2)\big]$$

matched precisely to gauge theory state

Slow string limit: $w \ll m$, i.e. $J \ll \sqrt{\lambda} m$

$$E = \sqrt{\lambda}\, m + ...$$

Very low string limit: $J = m \gg 1, \quad w = \frac{m}{\sqrt{\lambda}} \ll 1$

$$E = J\sqrt{\lambda + 1} = \sqrt{\lambda}J + ...$$

should correspond to antiferromagnetic state spin state

Quantum string corrections suppressed by $\frac{1}{\sqrt{\lambda}}$

but subleading terms in classical $E$ may receive corrections from higher loop orders

# Some conclusions

- Matching between gauge and string states near and far from BPS limit
- Presence of non-analytic in $\lambda$ terms in quantum string semiclassical expansion explains "3-loop" disagreement
- Implies presence of non-trivial $\lambda$-dependent phase in string S-matrix or in string Bethe ansatz
- Landau-Lifshitz action: low-energy effective action for relevant string/spin chain modes
- Quantization of Landau-Lifshitz action: effective method of finding finite size corrections on spin chain side and $\alpha'$ quantum corrections on string side
- Suggests integrability extends to quantum level: local action that reproduces results of Bethe ansatz based on 2-body S-matrix