BLACK HOLES AND QUBITS

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Abstract

We establish a correspondence between the entanglement measures of qubits in quantum information theory and the Bekenstein-Hawking entropy of black holes in string theory.


1) BLACK HOLES AND ENTANGLEMENT
Quantum entanglement lies at the heart of quantum information theory, with applications to quantum computing, teleportation, cryptography and communication. In the apparently separate world of quantum gravity, the Bekenstein-Hawking entropy of black holes has also occupied center stage.

Despite their apparent differences, recent work has established a correspondence between the tripartite entanglement measure of three qubits and the macroscopic entropy of the four-dimensional 8-charge $STU$ black hole of $N = 2$ supergravity.
The measure of tripartite entanglement of three qubits (Alice, Bob and Charlie), known as the 3-tangle $\tau_{ABC}$, and the entropy $S$ of the 8-charge $STU$ black hole of supergravity are related by:

$$S = \frac{\pi}{2} \sqrt{\tau_{ABC}}$$

Duff: hep-th/0601134
Further developments

- Further papers have written a more complete dictionary, which translates a variety of phenomena in one language to those in the other:

- For example, one can relate the classification of three-qubit entanglements to the classification of supersymmetric black holes as in the following table:
Black holes and entanglement

Qubits • STU black holes • $N = 8$ case • Wrapped D3-branes and 3 qubits • Wrapped M2-branes and

Table

<table>
<thead>
<tr>
<th>Class</th>
<th>$S_A$</th>
<th>$S_B$</th>
<th>$S_C$</th>
<th>Det $a$</th>
<th>Black hole</th>
<th>Susy</th>
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</tbody>
</table>

Table: Classification of three-qubit entanglements and their corresponding $D = 4$ black holes.
Further developments continued

The attractor mechanism on the black hole side is related to optimal local distillation protocols on the QI side; the supersymmetric and non-supersymmetric cases corresponding to the suppression or non-suppression of bit-flip errors.
Abstract

- There is also quantum information theoretic interpretation of the 56 charge $N = 8$ black hole in terms of a Hilbert space consisting of seven copies of the three-qubit Hilbert space.

- It relies on the decomposition $E_7(7) \supset [SL(2)]^7$ and admits the interpretation, via the Fano plane, of a tripartite entanglement of seven qubits, with the entanglement measure given by Cartan’s quartic $E_7(7)$ invariant.

- Since the Fano plane provides the multiplication table of the octonions, this means that the octonions, often written off as a lost cause in physics (Penrose, Streater), may actually be testable in the laboratory.
Nevertheless, we still do not know whether there are any physical reasons underlying these mathematical coincidences.

With this in mind, we also turn our attention to connecting the qubits to the *microscopic* origin of the black hole entropy.
Further developments continued

- We consider the configurations of intersecting D3-branes, whose wrapping around the six compact dimensions $T^6$ provides the microscopic string-theoretic interpretation of the charges, and associate the three-qubit basis vectors $|ABC\rangle$, $(A, B, C = 0$ or $1)$ with the corresponding 8 wrapping cycles.

- To wrap or not to wrap: that is the qubit
Further developments continued

- In particular, we relate a well-known fact of quantum information theory, that the most general real three-qubit state can be parameterized by four real numbers and an angle, to a well-known fact of string theory, that the most general STU black hole can be described by four D3-branes intersecting at an angle.
2) QUBITS
Two qubits

- The two qubit system Alice and Bob (where $A, B = 0, 1$) is described by the state
  \[
  |\psi\rangle = a_{AB}|AB\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.
  \]

- The bipartite entanglement of Alice and Bob is given by
  \[
  \tau_{AB} = 4|\det \rho_A| = 4|\det a_{AB}|^2,
  \]
  where
  \[
  \rho_A = Tr_B|\psi\rangle\langle\psi|
  \]
  It is invariant under $SL(2)_A \times SL(2)_B$, with $a_{AB}$ transforming as a $(2, 2)$, and under a discrete duality that interchanges A and B.
Two qubits

- Example, separable state:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle \]

\[ \tau_{AB} = 0 \]

- Example, Bell state:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]

\[ \tau_{AB} = 1 \]

- EPR “paradox”
The three qubit system Alice, Bob and Charlie (where $A, B, C = 0, 1$) is described by the state

$$|\psi\rangle = a_{ABC}|ABC\rangle$$

$$= a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle.$$
The tripartite entanglement of Alice, Bob and Charlie is given by

$$\tau_{ABC} = 4|\text{Det} \; a_{ABC}|,$$


**Det** $a_{ABC}$ is Cayley’s hyperdeterminant

$$\text{Det} \; a_{ABC} = -\frac{1}{2} \varepsilon A_1 A_2 \varepsilon B_1 B_2 \varepsilon C_1 C_4 \varepsilon C_2 C_3 \varepsilon A_3 A_4 \varepsilon B_3 B_4$$

$$\cdot a_{A_1 B_1 C_1} a_{A_2 B_2 C_2} a_{A_3 B_3 C_3} a_{A_4 B_4 C_4}$$
Symmetry

- Explicitly

\[
\text{Det } a_{ABC} =
\]
\[
a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\
- 2(a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} \\
+ a_{000} a_{100} a_{011} a_{111} + a_{001} a_{010} a_{101} a_{110} \\
+ a_{001} a_{100} a_{011} a_{110} + a_{010} a_{100} a_{011} a_{101}) \\
+ 4(a_{000} a_{011} a_{101} a_{110} + a_{001} a_{010} a_{100} a_{111}).
\]

- It is invariant under \( SL(2)_A \times SL(2)_B \times SL(2)_C \), with \( a_{ABC} \) transforming as \((2, 2, 2)\), and under a discrete triality that interchanges A, B and C.
Another useful quantity is the local entropy $S_A$, which is a measure of how entangled $A$ is with the pair $BC$:

$$S_A = 4 \det \rho_A \equiv \tau_{A(BC)}$$

where $\rho_A$ is the reduced density matrix

$$\rho_A = \text{Tr}_{BC} |\psi\rangle \langle \psi|,$$

and with similar formulae for $B$ and $C$. 
2-tangles $\tau_{AB}$, $\tau_{BC}$, and $\tau_{CA}$ give bipartite entanglements between pairs in 3-qubit system.

3-tangle $\tau_{ABC}$ is a measure of the genuine 3-way entanglement:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{CA}$$
### Entanglement classes

<table>
<thead>
<tr>
<th>Class</th>
<th>$\tau_{A(BC)}$</th>
<th>$\tau_{B(AC)}$</th>
<th>$\tau_{(AB)C}$</th>
<th>$\tau_{ABC}$</th>
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<td>A-BC</td>
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</tr>
<tr>
<td>B-CA</td>
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<td>&gt; 0</td>
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<tr>
<td>C-AB</td>
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<td>&gt; 0</td>
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<td>&gt; 0</td>
<td>&gt; 0</td>
<td>$\neq 0$</td>
</tr>
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</table>
Complex qubit parameters

- Two states of a composite quantum system are regarded as equivalent if they are related by a unitary transformation which factorizes into separate transformations on the component parts, so-called local unitaries. The Hilbert space decomposes into equivalence classes, or orbits under the action of the group of local unitaries.
For unnormalized three-qubit states, the number of parameters \cite{Linden and Popescu: quant-ph/9711016} needed to describe inequivalent states or, what amounts to the same thing, the number of algebraically independent invariants \cite{Sudbery: quant-ph/0001116} is given by the dimension of the space of orbits

\[
\frac{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}{U(1) \times SU(2) \times SU(2) \times SU(2)}
\]

namely, \(16 - 10 = 6\).
However, for subsequent comparison with the STU black hole [Duff, Liu and Rahmfeld: hep-th/9508094; Behrndt et al: hep-th/9608059], we restrict our attention to states with real coefficients $a_{ABC}$.

In this case, one can show that there are five algebraically independent invariants: $\text{Det } a$, $S_A$, $S_B$, $S_C$ and the norm $\langle \psi | \psi \rangle$, corresponding to the dimension of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$

$$\frac{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2}{SO(2) \times SO(2) \times SO(2)}$$

namely, $8 - 3 = 5$. 
Hence, the most general real three-qubit state can be described by just five parameters.  

It may conveniently be written

$$|\psi\rangle = -N_3 \cos^2 \theta |001\rangle - N_2 |010\rangle + N_3 \sin \theta \cos \theta |011\rangle - N_1 |100\rangle - N_3 \sin \theta \cos \theta |101\rangle + (N_0 + N_3 \sin^2 \theta) |111\rangle.$$
Representatives

Representatives from each class are:

- Class A-B-C (product states):
  \[ N_0 |111\rangle. \]

- Classes A-BC, (bipartite entanglement):
  \[ N_0 |111\rangle - N_1 |100\rangle, \]
  and similarly B-CA, C-AB.

- Class W (maximizes bipartite entanglement):
  \[-N_1 |100\rangle - N_2 |010\rangle - N_3 |001\rangle.\]

- Class GHZ (genuine tripartite entanglement):
  \[ N_0 |111\rangle - N_1 |100\rangle - N_2 |010\rangle - N_3 |001\rangle. \]
3) STU BLACK HOLES
The STU model consists of $N = 2$ supergravity coupled to three vector multiplets interacting through the special Kahler manifold $[SL(2)/SO(2)]^3$:

$$S_{STU} = \frac{1}{16\pi G} \int e^{-\eta} \left[ R + \frac{1}{4} \left( \text{Tr} \left[ \partial M_T^{-1} \partial M_T \right] + \text{Tr} \left[ \partial M_U^{-1} \partial M_U \right] \right) \right] \star 1$$

$$+ \star d\eta \wedge d\eta - \frac{1}{2} \star H[3] \wedge H[3] - \frac{1}{2} \star F_{S[2]}^T \wedge (M_T \otimes M_U) F_{S[2]}$$

$$M_S = \frac{1}{\mathcal{S}(S)} \left( \begin{array}{cc} 1 & \mathcal{R}(S) \\ \mathcal{R}(S) & |S|^2 \end{array} \right) \quad \text{etc.}$$
STU parameters

A general static spherically symmetric black hole solution depends on 8 charges denoted $q_0, q_1, q_2, q_3, p^0, p^1, p^2, p^3$, but the generating solution depends on just $8 - 3 = 5$ parameters [Cvetic and Youm: hep-th/9512127; Cvetic and Hull: hep-th/9606193], after fixing the action of the isotropy subgroup $[SO(2)]^3$. 
Black hole entropy

The STU black hole entropy is a complicated function of the 8 charges:

\[
\left(\frac{S}{\pi}\right)^2 = -(p \cdot q)^2 \\
+ 4 \left[ (p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2) \\
+ q_0 p^1 p^2 p^2 - p^0 q_1 q_2 q_3 \right]
\]

Behrndt et al: hep-th/9608059
Qubit correspondence

- By identifying the 8 charges with the 8 components of the three-qubit hypermatrix $a_{ABC}$,

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} a_{000} \\ -a_{001} \\ -a_{010} \\ -a_{100} \\ a_{111} \\ a_{110} \\ a_{101} \\ a_{011} \end{bmatrix}$$

one finds that the black hole entropy is related to the 3-tangle as in

$$S = \pi \sqrt{|\text{Det} a_{ABC}|} = \frac{\pi}{2} \sqrt{\tau_{ABC}}$$

Duff: hep-th/0601134
Embeddings

The solution can usefully be embedded in

- $N = 4$ supergravity with symmetry $SL(2) \times SO(6, 22)$, the low-energy limit of the heterotic string compactified on $T^6$, where the charges transform as a $(2, 28)$.
- $N = 8$ supergravity with symmetry $E_7(7)$, the low-energy limit of the Type IIA or Type IIB strings, compactified on $T^6$ or M-theory on $T^7$, where the charges transform as a 56.

Remarkably, the same five parameters suffice to describe these 56-charge black holes.
Table: Classification of three-qubit entanglements and their corresponding $D = 4$ black holes.

<table>
<thead>
<tr>
<th>Class</th>
<th>$S_A$</th>
<th>$S_B$</th>
<th>$S_C$</th>
<th>Det $a$</th>
<th>Black hole</th>
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$N = 8$ CASE

4) $N = 8$ CASE
There is, in fact, a quantum information theoretic interpretation of the 56 charge $N = 8$ black hole in terms of a Hilbert space consisting of seven copies of the three-qubit Hilbert space. It relies on the decomposition $E_7(7) \supset [SL(2)]^7$. 
Decomposition of the 56

Under

\[ E_{7(7)} \supset \]

\[ SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F \times SL(2)_G \]

the 56 decomposes as

\[ 56 \rightarrow \]

\[ (2, 2, 1, 2, 1, 1, 1) \]
\[ + (1, 2, 2, 1, 2, 1, 1) \]
\[ + (1, 1, 2, 2, 1, 2, 1) \]
\[ + (1, 1, 1, 2, 2, 1, 2) \]
\[ + (2, 1, 1, 1, 2, 2, 1) \]
\[ + (1, 2, 1, 1, 1, 2, 2) \]
\[ + (2, 1, 2, 1, 1, 1, 2) \]
Seven qubits

- It admits the interpretation of a tripartite entanglement of seven qubits, Alice, Bob, Charlie, Daisy, Emma, Fred and George:

\[
|\psi\rangle = a_{ABD}|ABD\rangle + b_{BCE}|BCE\rangle + c_{CDF}|CDF\rangle + d_{DEG}|DEG\rangle + e_{EFA}|EFA\rangle + f_{FGB}|FGB\rangle + g_{GAC}|GAC\rangle
\]
The following diagram may help illustrate the tripartite entanglement between the 7 qubits.
The entanglement measure given by Cartan’s quartic $E_{7(7)}$ invariant.

$$I_4 = -\text{Tr}((xy)^2) + \frac{1}{4}\text{Tr}(xy)^2 - 4(\text{Pf}(x) + \text{Pf}(y))$$

$x^{IJ}$ and $y_{IJ}$ are again $8 \times 8$ antisymmetric charge matrices.

Duff and Ferrara: quant-ph/0609227

Levay: hep-th/0610314
\[ x^{I,J} = \]

\[
\begin{pmatrix}
0 & -a_{111} & -b_{111} & -c_{111} & -d_{111} & -e_{111} & -f_{111} & -g_{111} \\
-a_{111} & 0 & f_{001} & d_{100} & -c_{010} & g_{010} & -b_{100} & -e_{001} \\
-b_{111} & -f_{001} & 0 & g_{001} & e_{100} & -d_{010} & a_{010} & -c_{100} \\
-c_{111} & -d_{100} & -g_{001} & 0 & a_{001} & f_{100} & -e_{010} & b_{010} \\
-d_{111} & c_{010} & -e_{100} & -a_{001} & 0 & b_{001} & g_{100} & -f_{010} \\
-e_{111} & -g_{010} & d_{010} & -f_{100} & -b_{001} & 0 & c_{001} & a_{100} \\
-f_{111} & b_{100} & -a_{010} & e_{010} & -g_{100} & -c_{001} & 0 & d_{001} \\
g_{111} & e_{001} & c_{100} & -b_{010} & f_{010} & -a_{100} & -d_{001} & 0 \\
\end{pmatrix}
\]
\[
Y_{IJ} = \\
\begin{pmatrix}
0 & -a_{000} & -b_{000} & -c_{000} & -d_{000} & -e_{000} & -f_{000} & -g_{000} \\
-a_{000} & 0 & f_{110} & d_{011} & -c_{101} & g_{101} & -b_{011} & -e_{110} \\
-b_{000} & -f_{110} & 0 & g_{110} & e_{011} & -d_{101} & a_{101} & -c_{011} \\
-c_{000} & -d_{011} & -g_{110} & 0 & a_{110} & f_{011} & -e_{101} & b_{101} \\
-d_{000} & c_{101} & -e_{011} & -a_{110} & 0 & b_{110} & g_{011} & -f_{101} \\
e_{000} & -g_{101} & d_{101} & -f_{011} & -b_{110} & 0 & c_{110} & a_{011} \\
f_{000} & b_{011} & -a_{101} & e_{101} & -g_{011} & -c_{110} & 0 & d_{110} \\
g_{000} & e_{110} & c_{011} & -b_{101} & f_{101} & -a_{011} & -d_{110} & 0 \\
\end{pmatrix}
\]
Schematically,

\[ l_4 = a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \]
\[ + 2 \left[ a^2 b^2 + a^2 c^2 + a^2 d^2 + a^2 e^2 + a^2 f^2 + a^2 g^2 \right. \]
\[ + b^2 c^2 + b^2 d^2 + b^2 e^2 + b^2 f^2 + b^2 g^2 \]
\[ + c^2 d^2 + c^2 e^2 + c^2 f^2 + c^2 g^2 \]
\[ + d^2 e^2 + d^2 f^2 + d^2 g^2 \]
\[ \left. + e^2 f^2 + e^2 g^2 + f^2 g^2 \right] \]
\[ + 8 \left[ abce + bcdf + cdeg + defa + efgb + fgac + gabd \right], \]

where \( a^4 \) is Cayley’s hyperdeterminant etc.
Remarkably, because the generating solution depends on the same five parameters as the \textit{STU} model, its classification of states will exactly parallel that of the usual three qubits. Indeed, the Cartan invariant reduces to Cayley’s hyperdeterminant in a canonical basis.

\textit{Kallosh and Linde: hep-th/0602061}
An alternative description is provided by the Fano plane which has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

Fano plane
Octonions

The Fano plane also provides the multiplication for the imaginary octonions:

<table>
<thead>
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<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>D</td>
<td>G</td>
<td>−B</td>
<td>F</td>
<td>−E</td>
<td>−C</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>−D</td>
<td>E</td>
<td>A</td>
<td>−C</td>
<td>G</td>
<td>−F</td>
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</tr>
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<td>−G</td>
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<td>F</td>
<td>B</td>
<td>−D</td>
<td>A</td>
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<td>−G</td>
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<td>D</td>
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<td>−D</td>
<td>−B</td>
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</table>
5) WRAPPED D3-BRANES AND 3 QUBITS
Microscopic analysis

- This is not unique since there are many ways of embedding the \textit{STU} model in string/M-theory, but a useful one from our point of view is that of four D3-branes wrapping the (579), (568), (478), (469) cycles of $T^6$ with wrapping numbers $N_0, N_1, N_2, N_3$ and intersecting over a string. Klebanov and Tseytlin: hep-th/9604166

- The wrapped circles are denoted by a cross and the unwrapped circles by a nought as shown in the following table:
Table: Three qubit interpretation of the 8-charge $D = 4$ black hole from four D3-branes wrapping around the lower four cycles of $T^6$ with wrapping numbers $N_0, N_1, N_2, N_3$. 

| Macro Charges | Micro Charges          | $|ABC\rangle$ |
|---------------|------------------------|---------------|
| $p^0$         | 0                      | $|000\rangle$ |
| $q_1$         | 0                      | $|110\rangle$ |
| $q_2$         | $-N_3\sin \theta \cos \theta$ | $|101\rangle$ |
| $q_3$         | $N_3 \sin \theta \cos \theta$ | $|011\rangle$ |
| $q_0$         | $N_0 + N_3 \sin^2 \theta$ | $|111\rangle$ |
| $-p^1$        | $-N_3 \cos^2 \theta$   | $|001\rangle$ |
| $-p^2$        | $-N_2$                 | $|010\rangle$ |
| $-p^3$        | $-N_1$                 | $|100\rangle$ |
Fifth parameter

The fifth parameter $\theta$ is obtained by allowing the $N_3$ brane to intersect at an angle which induces additional effective charges on the $(579), (569), (479)$ cycles [Balasubramanian and Larsen: hep-th/9704143; Balasubramanian: hep-th/9712215; Bertolini and Trigiante: hep-th/0002191].

The microscopic calculation of the entropy consists of taking the logarithm of the number of microstates and yields the same result as the macroscopic one [Bertolini and Trigiante: hep-th/0008201].
Qubit interpretation

- To make the black hole/qubit correspondence we associate the three $T^2$ with the $SL(2)_A \times SL(2)_B \times SL(2)_C$ of the three qubits Alice, Bob, and Charlie. The 8 different cycles then yield 8 different basis vectors $|ABC\rangle$ as in the last column of the Table, where $|0\rangle$ corresponds to xo and $|1\rangle$ to ox.

- We see immediately that we reproduce the five parameter three-qubit state $|\Psi\rangle$:

$$
|\Psi\rangle = -N_3\cos^2\theta|001\rangle - N_2|010\rangle + N_3\sin\theta\cos\theta|011\rangle - N_1|100\rangle - N_3\sin\theta\cos\theta|101\rangle + (N_0 + N_3\sin^2\theta)|111\rangle.
$$

- Note from the Table that the GHZ state describes four D3-branes intersecting over a string, or groups of 4 wrapping cycles with just one cross in common.
IIA and IIB

- Performing a T-duality transformation, one obtains a Type IIA interpretation with zero D6-branes, $N_0$ D0-branes, $N_1, N_2, N_3$ D4-branes plus effective D2-brane charges, where $|0\rangle$ now corresponds to xx and $|1\rangle$ to oo.
6) WRAPPED M2-BRANES AND 2 QUTRITS
Qutrit interpretation

- All this suggests that the analogy between $D = 5$ black holes and three-state systems (0 or 1 or 2), known as qutrits [Duff and Ferrara: 0704.0507 [hep-th]], should involve the choice of wrapping a brane around one of three circles in $T^3$. This is indeed the case, with the number of qutrits being two.

- The two-qutrit system (where $A, B = 0, 1, 2$) is described by the state

  $$|\psi\rangle = a_{AB}|AB\rangle,$$

  and the Hilbert space has dimension $3^2 = 9$. 
2-tangle

- The bipartite entanglement of Alice and Bob is given by the 2-tangle
  \[ \tau_{AB} = 27 \det \rho_A = 27 |\det a_{AB}|^2, \]
  where \( \rho_A \) is the reduced density matrix
  \[ \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| . \]

- The determinant is invariant under \( SL(3)_A \times SL(3)_B \), with \( a_{AB} \) transforming as a \((3, 3)\), and under a discrete duality that interchanges A and B.
For subsequent comparison with the $D = 5$ black hole, we restrict our attention to unnormalized states with real coefficients $a_{AB}$.

There are three algebraically independent invariants: $\tau_{AB}$, $C_2$ (the sum of the squares of the absolute values of the minors of $\rho_{AB}$) and the norm $\langle \Psi | \Psi \rangle$, corresponding to the dimension of

$$\mathbb{R}^3 \times \mathbb{R}^3$$

$$\frac{SO(3) \times SO(3)}{SO(3) \times SO(3)}$$

namely, $9 - 6 = 3$. 
Hence, the most general two-qutrit state can be described by just three parameters, which may conveniently taken to be three real numbers $N_0, N_1, N_2$.

$$|\psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

A classification of two-qutrit entanglements, depending on the rank of the density matrix, is given in the following table:
### $D = 5$ table

<table>
<thead>
<tr>
<th>Class</th>
<th>$C_2$</th>
<th>$\tau_{AB}$</th>
<th>Black hole</th>
<th>Susy</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-B</td>
<td>0</td>
<td>0</td>
<td>small</td>
<td>$1/2$</td>
</tr>
<tr>
<td>Rank 2 Bell</td>
<td>$&gt;0$</td>
<td>0</td>
<td>small</td>
<td>$1/4$</td>
</tr>
<tr>
<td>Rank 3 Bell</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>large</td>
<td>$1/8$</td>
</tr>
</tbody>
</table>

**Table**: Classification of two-qutrit entanglements and their corresponding $D = 5$ black holes.
The 9-charge $N = 2, D = 5$ black hole may also be embedded in the $N = 8$ theory in different ways. The most convenient microscopic description is that of three M2-branes wrapping the (58), (69), (710) cycles of the $T^6$ compactification of $D = 11$ M-theory, with wrapping numbers $N_0, N_1, N_2$ and intersecting over a point [Papadopoulos and Townsend: hep-th/9603087; Klebanov and Tseytlin: hep-th/9604166].

To make the black hole/qutrit correspondence we associate the two $T^3$ with the $SL(3)_A \times SL(3)_B$ of the two qutrits Alice and Bob, where $|0\rangle$ corresponds to xoo, $|1\rangle$ to oxo and $|2\rangle$ to oox. The 9 different cycles then yield the 9 different basis vectors $|AB\rangle$ as in the last column of the following Table:
### $D = 5$ table

| 5 | 6 | 7 | 8 | 9 | 10 | macro charges | micro charges | $|AB\rangle$ |
|---|---|---|---|---|---|--------------|--------------|----------|
| x | o | o | x | o | o | $p^0$ | $N_0$ | $|00\rangle$ |
| o | x | o | o | x | o | $p^1$ | $N_1$ | $|11\rangle$ |
| o | o | x | o | o | x | $p^2$ | $N_2$ | $|22\rangle$ |
| x | o | o | o | x | o | $p^3$ | 0 | $|01\rangle$ |
| o | x | o | o | o | x | $p^4$ | 0 | $|12\rangle$ |
| o | o | x | x | o | o | $p^5$ | 0 | $|20\rangle$ |
| x | o | o | o | o | x | $p^6$ | 0 | $|02\rangle$ |
| o | x | o | x | o | o | $p^7$ | 0 | $|10\rangle$ |
| o | o | x | o | x | o | $p^8$ | 0 | $|21\rangle$ |
We see immediately that we reproduce the three parameter two-qutrit state $|\psi\rangle$:

$$|\psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

The black hole entropy, both macroscopic and microscopic, turns out to be given by the 2-tangle

$$S = 2\pi \sqrt{|\det a_{AB}|},$$

and the classification of the two-qutrit entanglements matches that of the black holes.

Note that the non-vanishing cubic combinations appearing in $\det a_{AB}$ correspond to groups of 3 wrapping cycles with no crosses in common, i.e. that intersect over a point.
Embeddings

- There is, in fact, a quantum information theoretic interpretation of the 27 charge $N = 8, D = 5$ black hole in terms of a Hilbert space consisting of three copies of the two-qutrit Hilbert space. It relies on the decomposition $E_{6(6)} \supset [SL(3)]^3$ and admits the interpretation of a bipartite entanglement of three qutrits, with the entanglement measure given by Cartan’s cubic $E_{6(6)}$ invariant. Duff and Ferrara: 0704.0507 [hep-th]

- Once again, however, because the generating solution depends on the same three parameters as the 9-charge model, its classification of states will exactly parallel that of the usual two qutrits. Indeed, the Cartan invariant reduces to $\det a_{AB}$ in a canonical basis. Ferrara and Maldacena: hep-th/9706097
SUMMARY
Summary

- Our Type IIB microscopic analysis of the $D = 4$ black hole has provided an explanation for the appearance of the qubit two-valuedness (0 or 1) that was lacking in the previous treatments: The brane can wrap one circle or the other in each $T^2$.

- The number of qubits is three because of the number of extra dimensions is six.

- The five parameters of the real three-qubit state are seen to correspond to four D3-branes intersecting at an angle.
Summary

- Our M-theory analysis of the $D = 5$ black hole has provided an explanation for the appearance of the qutrit three-valuedness ($0$ or $1$ or $2$) that was lacking in the previous treatments: The brane can wrap one of the three circles in each $T^3$.

- The number of qutrits is two because of the number of extra dimensions is six.

- The three parameters of the real two-qutrit state are seen to correspond to three intersecting M2-branes.
Summary

- It would be interesting to see whether we can now find an underlying physical justification