Argyres-Seiberg Duality and New SCFT’s

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w/ Philip Argyres and Paul Esposito
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Argyres-Seiberg Duality

Philip Argyres & Nathan Seiberg 0711.0054

$$\mathfrak{g}[\{d_i\}] \text{ w/ } r \simeq \tilde{\mathfrak{g}}[\{\tilde{d}_i\}] \text{ w/ } (\tilde{r} \oplus \text{SCFT}[d : \mathfrak{h}])$$

- LHS: There is a gauge group $\mathfrak{g}$ with matter charged in representations $r$.

- RHS: There is a rank 1 SCFT with mass dimension of the Coulomb branch moduli $d$ and flavor symmetry $\mathfrak{h}$. Then $\tilde{\mathfrak{g}} \subset \mathfrak{h}$ is gauged with matter charged in representations $\tilde{r}$. 
II Criteria for Duality

Philip Argyres & JRW 0712.2028

• The spectrum of dimensions of Coulomb branch vevs:
  \[ \{d_i\} = \{\tilde{d}_i\} \cup \{d\}. \]

• The flavor symmetry algebras:
  \[ f = \tilde{f} \oplus H. \]

• The beta function from weakly gauging the flavor symmetry:
  \[ T(r) = T(\tilde{r}) + k_h \cdot I_{H \hookrightarrow h}. \]

• The number of marginal couplings:
  \[ 2 \cdot T(\tilde{g}) = T(\tilde{r}) + k_{\tilde{g}} \cdot I_{\tilde{g} \hookrightarrow h}. \]
Criteria for Duality (cont’d)

• The contribution to the $u(1)_R$ central charge (for the SCFT):
  \[ \frac{3}{2} \cdot k_R = 24 \cdot c = 4 \cdot (|g| - |\tilde{g}|) + (|r| - |\tilde{r}|). \]

• The contribution to the $a$ conformal anomaly (for the SCFT):
  \[ 48 \cdot a = 10 \cdot (|g| - |\tilde{g}|) + (|r| - |\tilde{r}|). \]

• The existence of a global $\mathbb{Z}_2$-obstruction to gauging the flavor symmetry.
Criteria for Duality ($a$ and $c$ anomalies)

- In Lagrangian theories, the $a$ and $c$ anomalies can be computed by t' Hooft anomaly matching:
  \[ 4 \cdot (2 \cdot a - c) = |g| = \sum_i (2 \cdot d_i - 1). \]

- If we look back at the criteria the SCFT satisfies a similar relation:
  \[ 4 \cdot (2 \cdot a - c) = (2 \cdot d - 1). \]

- Shapere and Tachikawa have given a proof that this formula holds for a large class of $N = 2$ SCFT's in 0804.1957.
Criteria for Duality ($\mathbb{Z}_2$-obstruction)


$G_2 \ w/ \ 8 \cdot 7 \cong su(2) \ w/ \ (2 \oplus SCFT[6:sp(5)])$

- **LHS:** The 7 of $G_2$ is real $\Rightarrow$ the flavor symmetry is sp(4) when the sp(4) is gauged there is a $\mathbb{Z}_2$-obstruction because the 8 is pseudoreal.

- **RHS:** The su(2) has an anomaly related to the single 2 $\Rightarrow$ sp(5) must posses a $\mathbb{Z}_2$-obstruction to gauging in order to cancel this since the LHS is anomaly free $\Rightarrow$ this gives sp(4) a $\mathbb{Z}_2$-obstruction since $I_{f \hookrightarrow sp(5)} = 1$ for $f$ either su(2) or sp(4).
### Examples of Duality and Results (1 Marginal Operator)

<table>
<thead>
<tr>
<th></th>
<th>$g$</th>
<th>$w/r$</th>
<th>$\tilde{g}$</th>
<th>$\tilde{r}$</th>
<th>SCFT</th>
<th>$d/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$sp(3)$</td>
<td>$14 \oplus 11 \cdot 6$</td>
<td>$sp(2)$</td>
<td></td>
<td></td>
<td>$6 : E_8$</td>
</tr>
<tr>
<td>2</td>
<td>$su(6)$</td>
<td>$20 \oplus 15 \oplus 15 \cdot 5 \cdot 6 \oplus 5 \cdot 6 \cdot 7$</td>
<td>$su(5)$</td>
<td>$5 \oplus 3 \oplus 10 \oplus 10$</td>
<td></td>
<td>$6 : E_8$</td>
</tr>
<tr>
<td>3</td>
<td>$so(12)$</td>
<td>$3 \cdot 32 \oplus 32' \oplus 4 \cdot 12$</td>
<td>$so(11)$</td>
<td>$3 \cdot 32$</td>
<td></td>
<td>$6 : E_8$</td>
</tr>
<tr>
<td>4</td>
<td>$G_2$</td>
<td>$8 \cdot 7$</td>
<td>$su(2)$</td>
<td>$2$</td>
<td></td>
<td>$6 : sp(5)$</td>
</tr>
<tr>
<td>5</td>
<td>$so(7)$</td>
<td>$4 \cdot 8 \oplus 6 \cdot 7$</td>
<td>$sp(2)$</td>
<td>$5 \cdot 4$</td>
<td></td>
<td>$6 : sp(5)$</td>
</tr>
<tr>
<td>6</td>
<td>$su(6)$</td>
<td>$21 \oplus 21 \oplus 20 \oplus 6 \oplus 6$</td>
<td>$su(5)$</td>
<td>$10 \oplus 10$</td>
<td></td>
<td>$6 : sp(5)$</td>
</tr>
<tr>
<td>7</td>
<td>$sp(2)$</td>
<td>$12 \cdot 4$</td>
<td>$su(2)$</td>
<td></td>
<td></td>
<td>$4 : E_7$</td>
</tr>
<tr>
<td>8</td>
<td>$su(4)$</td>
<td>$2 \cdot 6 \oplus 6 \cdot 4 \oplus 6 \cdot 4$</td>
<td>$su(3)$</td>
<td>$2 \cdot 3 \oplus 2 \cdot 3$</td>
<td></td>
<td>$4 : E_7$</td>
</tr>
<tr>
<td>9</td>
<td>$so(7)$</td>
<td>$6 \cdot 8 \oplus 4 \cdot 7$</td>
<td>$G_2$</td>
<td>$4 \cdot 7$</td>
<td></td>
<td>$4 : E_7$</td>
</tr>
<tr>
<td>10</td>
<td>$so(8)$</td>
<td>$6 \cdot 8 \oplus 4 \cdot 8' \oplus 2 \cdot 8''$</td>
<td>$so(7)$</td>
<td>$6 \cdot 8$</td>
<td></td>
<td>$4 : E_7$</td>
</tr>
<tr>
<td>11</td>
<td>$so(8)$</td>
<td>$6 \cdot 8 \oplus 6 \cdot 8'$</td>
<td>$G_2$</td>
<td></td>
<td></td>
<td>$4 : E_7 \oplus [4 : E_7]$</td>
</tr>
<tr>
<td>12</td>
<td>$sp(2)$</td>
<td>$6 \cdot 5$</td>
<td>$su(2)$</td>
<td></td>
<td></td>
<td>$4 : sp(3) \oplus su(2)$</td>
</tr>
<tr>
<td>13</td>
<td>$sp(2)$</td>
<td>$4 \cdot 4 \oplus 4 \cdot 5$</td>
<td>$su(2)$</td>
<td>$3 \cdot 2$</td>
<td></td>
<td>$4 : sp(3) \oplus su(2)$</td>
</tr>
<tr>
<td>14</td>
<td>$su(4)$</td>
<td>$10 \oplus 10 \oplus 2 \cdot 4 \oplus 2 \cdot 4$</td>
<td>$su(3)$</td>
<td>$3 \oplus 3$</td>
<td></td>
<td>$4 : sp(3) \oplus su(2)$</td>
</tr>
<tr>
<td>15</td>
<td>$su(3)$</td>
<td>$6 \cdot 3 \oplus 6 \cdot 3$</td>
<td>$su(2)$</td>
<td>$2 \cdot 2$</td>
<td></td>
<td>$3 : E_6$</td>
</tr>
<tr>
<td>16</td>
<td>$su(4)$</td>
<td>$4 \cdot 6 \oplus 4 \cdot 4 \oplus 4 \cdot 4$</td>
<td>$sp(2)$</td>
<td>$6 \cdot 4$</td>
<td></td>
<td>$3 : E_6$</td>
</tr>
<tr>
<td>17</td>
<td>$su(3)$</td>
<td>$3 \oplus 3 \oplus 6 \oplus 6$</td>
<td>$su(2)$</td>
<td>$n \cdot 2$</td>
<td></td>
<td>$3 : h$</td>
</tr>
</tbody>
</table>

- predicted dualities with one marginal operator
Examples of Duality and Results (2 Marginal Operators)

<table>
<thead>
<tr>
<th>(g)</th>
<th>(w/) (r)</th>
<th>= (\tilde{g}) (w/) (\tilde{r}) (\oplus) SCFT ([d : h])</th>
</tr>
</thead>
<tbody>
<tr>
<td>18. (su(2) \oplus su(3))</td>
<td>(2 \cdot (2, 1) \oplus (2, 3 \oplus \bar{3}) \oplus 4 \cdot (1, 3 \oplus \bar{3}))</td>
<td>(su(2) \oplus su(2)) (2 \cdot (2, 1) \oplus 2 \cdot (1, 2)) ([3 : E_6])</td>
</tr>
<tr>
<td>19. (su(2) \oplus sp(2))</td>
<td>(2 \cdot (2, 4) \oplus 8 \cdot (1, 4))</td>
<td>(su(2) \oplus su(2))</td>
</tr>
<tr>
<td>20. (su(2) \oplus sp(2))</td>
<td>(3 \cdot (2, 1) \oplus (2, 5) \oplus 4 \cdot (1, 5))</td>
<td>(su(2) \oplus su(2)) (3 \cdot (2, 1)) ([4 : sp(3) \oplus su(2)])</td>
</tr>
<tr>
<td>21. (su(2) \oplus G_2)</td>
<td>((2, 1) \oplus (2, 7) \oplus 6 \cdot (1, 7))</td>
<td>(su(2) \oplus su(2)) ((2, 1) \oplus (1, 1)) ([6 : sp(5)])</td>
</tr>
<tr>
<td>22. (su(3) \oplus su(3))</td>
<td>(2 \cdot (3, 3) \oplus 2 \cdot (3, 3))</td>
<td>(su(2) \oplus su(3)) (2 \cdot (2, 1)) ([3 : E_6])</td>
</tr>
<tr>
<td>23. (su(3) \oplus su(3))</td>
<td>((3 \oplus 3, 3 \oplus 3))</td>
<td>(su(2) \oplus su(3)) (2 \cdot (2, 1)) ([3 : E_6])</td>
</tr>
<tr>
<td>24. (su(3) \oplus su(3))</td>
<td>(3 \cdot (3 \oplus 3, 1) \oplus (3 \oplus 3) \oplus (\bar{3}, 3))</td>
<td>(su(2) \oplus su(3)) (2 \cdot (2, 1)) ([3 : E_6])</td>
</tr>
<tr>
<td>25. (su(3) \oplus sp(2))</td>
<td>((3 \oplus 3, 1) \oplus (3 \oplus 3, 5))</td>
<td>(su(2) \oplus sp(2)) (2 \cdot (2, 1)) ([3 : E_6])</td>
</tr>
<tr>
<td>26. (su(3) \oplus sp(2))</td>
<td>((3 \oplus 3, 1) \oplus (3 \oplus 3, 4) \oplus 6 \cdot (1, 4))</td>
<td>(su(2) \oplus sp(2)) (2 \cdot (2, 1) \oplus 6 \cdot (1, 4)) ([3 : E_6])</td>
</tr>
<tr>
<td>27. (sp(2) \oplus sp(2))</td>
<td>(2 \cdot (5, 1) \oplus (5, 4) \oplus 7 \cdot (1, 4))</td>
<td>(su(2) \oplus sp(2)) (7 \cdot (1, 4)) ([4 : sp(3) \oplus su(2)])</td>
</tr>
<tr>
<td>28. (sp(2) \oplus sp(2))</td>
<td>(4 \cdot (4, 1) \oplus 2 \cdot (4, 4) \oplus 4 \cdot (1, 4))</td>
<td>(su(2) \oplus sp(2)) (4 \cdot (1, 4)) ([4 : E_7])</td>
</tr>
<tr>
<td>29. (sp(2) \oplus G_2)</td>
<td>(5 \cdot (4, 1) \oplus (4, 7) \oplus 4 \cdot (1, 7))</td>
<td>(su(2) \oplus G_2) (4 \cdot (1, 7)) ([4 : E_7])</td>
</tr>
</tbody>
</table>

- predicted dualities with two marginal operators
Examples of Duality and Results (New SCFT’s)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\mathfrak{h}$</th>
<th>$k_{\mathfrak{h}}$</th>
<th>$3/2 \cdot k_R$</th>
<th>$48 \cdot a$</th>
<th>$\mathbb{Z}_2$ anomaly?</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$E_8$</td>
<td>12</td>
<td>124</td>
<td>190</td>
<td>no</td>
</tr>
<tr>
<td>6</td>
<td>$\text{sp}(5)$</td>
<td>7</td>
<td>98</td>
<td>164</td>
<td>yes</td>
</tr>
<tr>
<td>4</td>
<td>$E_7$</td>
<td>8</td>
<td>76</td>
<td>118</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>$\text{sp}(3) \oplus \text{su}(2)$</td>
<td>$5 \oplus 8$</td>
<td>58</td>
<td>100</td>
<td>yes $\oplus$ no</td>
</tr>
<tr>
<td>3</td>
<td>$E_6$</td>
<td>6</td>
<td>52</td>
<td>82</td>
<td>no</td>
</tr>
<tr>
<td>3</td>
<td>$2 \leq \text{rank}(\mathfrak{h}) \leq 6$</td>
<td>$(8 - n)/I_{\text{su}(2) \hookrightarrow \mathfrak{h}}$</td>
<td>$38 - 2n$</td>
<td>$68 - 2n$</td>
<td>?</td>
</tr>
</tbody>
</table>

- From arguments found in 0712.2028 we can restrict $\text{rank}(\mathfrak{h}) = 2$ which requires $n = 2$ in order to match the flavor symmetries.

- The flavor central charges, $k_{\mathfrak{h}}$, were confirmed for $E_6$, $E_7$, and $E_8$ through an F-theory calculation by Aharony and Tachikawa in 0711.4532.
The physics is encoded by:

- The Seiberg-Witten curve:
  \[ y^2 = x^3 + f(u, m_i)x + g(u, m_i) \]

- and the Seiberg-Witten 1-form: \( \lambda_{SW} \)
  \[ \partial_u \lambda_{SW} = \frac{dx}{y} + \partial_x(\ast)dx. \]

The charged states of the theory are encoded by:

- \( u(1) \) charges of a physical state are given by the homology class of a cycle, \( \gamma = n_e[\alpha] + n_m[\beta] \) (when \( m_i = 0 \)).

- These states have central charge, \( Z = \oint_{\gamma} \lambda_{SW} \).
Seiberg-Witten Theory (Singularities)

The singularities of the Seiberg-Witten curve:

- are located at $\Delta = 4 \cdot f^3 + 27 \cdot g^2 = 0$.

- If $m_i = 0$ then $\Delta \sim u^n$.

- The singularities physically correspond to a breakdown of the low-energy description $\Rightarrow$ charged states are becoming massless at this point in moduli space.
Seiberg-Witten Theory (Singularities with $m_i = 0$)
Seiberg-Witten Theory ($m_i \neq 0$)

- When mass parameters are turned on they appear in the curve in the form of invariants of the Weyl group of the flavor symmetry.

$$\Delta = u^n + P_D(u)\{m_i\} u^{n-1} + \ldots + P_{nD}(u)\{m_i\}$$

- The factorization of $\Delta$ is related to the flavor symmetry group through the fact that different flavor symmetries $\leftrightarrow$ different factorizations of $\Delta$. 

Seiberg-Witten Theory (Singularities with $m_i \neq 0$)
V The Kodaira Classification

• Kodaira classified the degenerations of holomorphic families of elliptic curves over one variable, $u$.

• The classification is of singularities that do not degenerate the holomorphic 1-form at the singularity.

• Fixing the holomorphic 1-form, $\omega = \frac{dx}{y}$, and requiring the singularities occur as $u \to 0$ specifies the curves exactly up to overall rescalings of $u$. 
The Kodaira Classification

Recall:

- \[ y^2 = x^3 + f(u)x + g(u) \]
- \[ \partial_u \lambda_{SW} = \frac{dx}{y} + \partial_x(\star)dx \]
- \[ Z = \oint \gamma \lambda_{SW} \]

It is easy to reproduce Kodaira’s classification by a little algebra. For details see sections 2.2 & 2.3 of hep-th/0504070 by Philip Argyres, Michael Crescimanno, Alfred Shapere, and JRW.
The Kodaira Classification

<table>
<thead>
<tr>
<th>name</th>
<th>curve</th>
<th>$\Delta_x \propto$</th>
<th>$D(u)$</th>
<th>$D(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_8$</td>
<td>$y^2 = x^3 + 2u^5$</td>
<td>$u^{10}$</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$y^2 = x^3 + u^3x$</td>
<td>$u^9$</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$y^2 = x^3 + u^4$</td>
<td>$u^8$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$y^2 = x^3 + 3\tau u^2 x + 2u^3$</td>
<td>$u^6(\tau^3 + 1)$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$y^2 = x^3 + u^2$</td>
<td>$u^4$</td>
<td>3/2</td>
<td>1</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$y^2 = x^3 + ux$</td>
<td>$u^3$</td>
<td>4/3</td>
<td>2/3</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$y^2 = x^3 + u$</td>
<td>$u^2$</td>
<td>6/5</td>
<td>2/5</td>
</tr>
<tr>
<td>$D_{n&gt;4}$</td>
<td>$y^2 = x^3 + 3ux^2 + 4\Lambda^{-2(n-4)}u^{n-1}$</td>
<td>$u^{n+2}(1 + \Lambda^{-2(n-4)}u^{n-4})$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$A_{n\geq0}$</td>
<td>$y^2 = (x - 1)(x^2 + \Lambda^{-(n+1)}u^{n+1})$</td>
<td>$u^{n+1}(1 + \Lambda^{-(n+1)}u^{n+1})$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- The result is two infinite series and seven "exceptional" curves.
- $\Lambda$ is the UV strong coupling scale and $\tau$ is the marginal gauge coupling.
The Kodaira Classification (Complex Deformations)

- The general mass deformations of these curves correspond to complex structure deformations that are subleading singularities as $u \to 0$.

<table>
<thead>
<tr>
<th>$E_8$</th>
<th>$y^2 = x^3 + x(M_2 u^3 + M_8 u^2 + M_{14} u + M_{20}) + (2 u^5 + M_{12} u^3 + M_{18} u^2 + M_{24} u + M_{30})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_7$</td>
<td>$y^2 = x^3 + x(u^3 + M_8 u + M_{12}) + (M_2 u^4 + M_6 u^3 + M_{10} u^2 + M_{14} u + M_{18})$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$y^2 = x^3 + x(M_2 u^2 + M_5 u + M_8) + (u^4 + M_6 u^2 + M_9 u + M_{12})$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$y^2 = x^3 + x(3 \tau u^2 + M_2 u + M_4) + (2 u^3 + \tilde{M}_4 u + M_6)$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$y^2 = x^3 + x(M_{1/2} u + M_2) + (u^2 + M_3)$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$y^2 = x^3 + x(u) + (M_{2/3} u + M_2)$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$y^2 = x^3 + x(M_{4/5}) + (u)$</td>
</tr>
<tr>
<td>$D_{n&gt;4}$</td>
<td>$y^2 = x^3 + 3 u x^2 + \Lambda^{-(n-4)} \tilde{M}<em>n x + 4 \Lambda^{-2(n-4)}(u^{n-1} + M_2 u^{n-2} + \cdots + M</em>{2n-2})$</td>
</tr>
<tr>
<td>$A_{n\geq 0}$</td>
<td>$y^2 = (x - 1)(x^2 + \Lambda^{-(n+1)}[u^{n+1} + M_2 u^{n-1} + M_3 u^{n-2} + \cdots + M_{n+1}])$</td>
</tr>
</tbody>
</table>

(2)
Kodaira Classification (The $A_{n>0}$ series)

- The curve shown corresponds to a $u(1)$ gauge theory with $n+1$ hypermultiplets all of the same charge, $\pm 1$.

- The beta function determines the form of the singularity. Let there be $n_a$ equal mass hypermultiplets with charge $\pm r_a$ then $b = \sum_a n_a r_a^2$.

- $b = n + 1 \rightarrow A_n$ singularity.

- $b = \sum_a n_a r_a^2 \rightarrow \oplus_a u(n_a)$ flavor symmetry.

- Since $b > 0$ these theories are all IR free.

- This is an example of the theme, singularity $\Leftrightarrow$ gauge group.
The Kodaira Classification (The $D_n$ series)

- The curve (for $n > 4$) written corresponds to an $\text{su}(2)$ gauge theory with $2n$ half-hypers in the fundamental representation $\Rightarrow b = 2(n - 4)$.

- Again $b > 0$ so all these theories are IR free.

- There are two types of representations for $\text{su}(2)$, the real $2r + 1$ and the pseudoreal $2s$.

- To avoid anomalies we must have $2n_r$ of each real representation and any number $m_s$ of the pseudoreal such that $\frac{1}{3}\sum_s m_s s(4s^2 - 1)$ is even.

- $b = \frac{4}{3}\sum_r n_r r(r + 1)(2r + 1) + \frac{1}{3}\sum_s m_s s(4s^2 - 1) - 8$

- The flavor symmetry that corresponds to this value of the beta function is $\oplus_r \text{sp}(n_r) \oplus_s \text{so}(m_s)$.
The Kodaira Classification (Vanishing beta function)

There are two ways to make $b = 0$ for the $su(2)$ beta function.

- The first case is $m_1 = 8$ and all other zero. This has a flavor symmetry of $so(8)$.

- This curve is the fully mass deformed $D_4$ curve.

- The second case is $n_1 = 1$ and all other zero. This has a flavor symmetry of $sp(1)$ and enhances the susy to $N = 4$.

- The curve for the second case is $y^2 = \prod_i(x - e_iu - e_i^2 M_2)$.

- $\Delta = \prod_{i<j}(e_i - e_j)^2(u + (e_i + e_j)M_2)^2$
The Kodaira Classification (Asymptotically free (or AF) theories)

These come from looking at su(2) gauge theories with $b < 0$.

- If we put in only fundamental matter: $b = m_1 - 8$.

- $m_1$ is the number of half-hypers and to avoid anomalies $m_1$ must be even $\Rightarrow m_1 = 2, 4, 6$.

- When all the masses are taken to be the same we get the $H_{1,2,3}$ mass deformed curves, respectively.
The Kodaira Classification ($E_{6,7,8}$ mass deformations)

The $E_{6,7,8}$ curves correspond to strongly interacting fixed points.

- There existence was suggested from stringy constructions.

- The maximal mass deformations were worked out by Minahan and Nemeschansky in:
  Nuclear Physics B 482 (1996) 142-152 and

- Evidence for the existence of new mass deformations was found by Philip Argyres & JRW in 0712.2028.
VI Central Charges and Curves

Shapere and Tachikawa 0804.1957

The twisted version of Seiberg-Witten theory relates the anomalies and central charges to:

- The mass dimension of the vev on moduli space,
- the # of neutral hypermultiplets on moduli space and
- the # of singular points of the Seiberg-Witten curve.
Central Charges and Curves (Twisted PI)

\[ \int \! [\! du \!][dq] \ A^\chi B^\sigma C^n e^{-S_{low\text{-}energy}} \]

- \( \chi \) is the Euler characteristic.
- \( \sigma \) is the signature.
- \( n \) is an instanton number.
- \( A^2 = \text{det} \left[ \frac{\partial u_i}{\partial a_j} \right] \)
- \( B^8 = \text{Radical} [\Delta] \)
Central Charges and Curves (master equations)

- The scaling behavior of the measure determines the R-charge of the states becoming massless at a singularity.

- The normalization is: \( R(\star) = 2 \cdot D(\star) \).

- \( 48 \cdot a = 12 \cdot R(A) + 8 \cdot R(B) + 10 \cdot r + 2 \cdot h \)

- \( 24 \cdot c = 8 \cdot R(B) + 4 \cdot r + 2 \cdot h \)

- \( 4(2 \cdot a - c) = 2 \cdot R(A) + r = \sum_{i=1}^{r} 2 \cdot (d_i - 1) + r = \sum_{i=1}^{r} (2 \cdot d_i - 1) \)

\( r \equiv \text{the complex dimension of moduli space} \)

\( h \equiv \text{the \# of massless neutral hypermultiplets} \)
Central Charges and Curves ($r = 1$)

- $R(A) = d - 1$
- $R(B) = \frac{1}{4} \cdot Z \cdot d$
- $Z \equiv$ The # of singular points of the Seiberg-Witten curve.
- $24 \cdot c = 2 \cdot Z \cdot d + 4 + 2 \cdot h$
- $k_{\hbar} = 2 \cdot d - h$
Central Charges and Curves (the unknown solution)

- $15 = 3 \cdot Z + h$

- $\frac{6}{I_{\text{su}(2) \hookrightarrow \hbar}} = 6 - h$

- The only rank 2 Lie Algebras are $\text{su}(2) \oplus \text{su}(2)$, $\text{su}(3)$, $\text{sp}(2)$, and $G_2$
Central Charges and Curves (Results)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\mathfrak{h}$</th>
<th>$Z$</th>
<th>$2 \cdot h$</th>
<th>rep.’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$E_8$</td>
<td>10</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>$\text{sp}(5)$</td>
<td>7</td>
<td>10</td>
<td>10(s)</td>
</tr>
<tr>
<td>4</td>
<td>$E_7$</td>
<td>9</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>$\text{sp}(3) \oplus \text{su}(2)$</td>
<td>6</td>
<td>(6,0)</td>
<td>$6 \oplus 1(s)$</td>
</tr>
<tr>
<td>3</td>
<td>$E_6$</td>
<td>8</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>rank($\mathfrak{h}$) = 2</td>
<td>4,5</td>
<td>6,0</td>
<td>?</td>
</tr>
</tbody>
</table>

Since there are no neutral hypermultiplets on the LHS of the equivalence $\Rightarrow$ the neutral hypermultiplets on the RHS must be charged under the flavor symmetry.
• There is a $\mathbb{Z}_2$-obstruction for the $\text{sp}(5)$ and $\text{sp}(3)$ factors.

• This obstruction comes from the neutral hypermultiplet charged in a pseudoreal representation.

• Consider our old example in this new light:

\[
G_2 \text{ w/ } 8 \cdot 7 \simeq \text{su}(2) \text{ w/ } (2 \oplus \text{SCFT}[6 : \text{sp}(5)]) \\
\text{su}(2) \oplus \text{sp}(4) \subset \text{sp}(5) \\
(2, 1) \oplus (1, 8) = 10
\]
VIII Constructing New Seiberg-Witten Curves

- The work of Shapere and Tachikawa specifies possible forms of the discriminant of the Seiberg-Witten curves.

- The discriminants are determined by partitioning the total order of the singularity at $m_i = 0$ into a $\mathbb{Z}$-tuple of integers.
Constructing New Seiberg-Witten Curves

There are 4 singular points and 8 singularities.

- $\Delta \sim (u + ...)^5(u^3 + ...)$
- $\Delta \sim (u + ...)^4(u + ...)^2(u^2 + ...)$
- $\Delta \sim (u + ...)^3(u + ...)(u^2 + ...)^2$
- $\Delta \sim (u^2 + ...)^3(u^2 + ...)$
- $\Delta \sim (u^4 + ...)^2$

A systematic search for su(3) reveals 2 solutions. We need to carry out a systematic search for su(2) $\oplus$ su(2), $G_2$, and sp(2).
Constructing New Seiberg-Witten Curves (1\textsuperscript{st} consistent su(3) solution)

- \( y^2 = x^3 + 3N_2x[u^2 + (1 + \nu)N_2^3 + N_3^2] + [u^4 + u^2((1 + 2\nu)N_2^3 + 2N_3^2) + \nu(1 + \nu)N_2^6 + (1 + 2\nu)N_2^3N_3^2 + N_3^4] \)

- \( \Delta = -27[u^2 + (1 + \nu)N_2^3 + N_3^2]^2[u^2 + (2 + \nu)N_2^3 + N_3^2]^2 \)

- Upon constructing the SW 1-form for this curve we find that it is impossible.
Constructing New Seiberg-Witten Curves (2\textsuperscript{nd} consistent su(3) solution)

\begin{itemize}
  \item $y^2 = x^3 + u[3N_2x(u - 4N_3) + u^3 - 12u^2N_3 - u(N_2^3 - 48N_3^2) - 64N_3^3]$
  \item $\Delta = -27u^2[u^3 - 12u^2N_3 + u(N_2^3 + 48N_3^2) - 64N_3^3]^2$
  \item When we compute the SW 1-form we find that it is identical zero. Therefore this is not a valid solution.
\end{itemize}
Constructing New Seiberg-Witten Curves

There are 5 singular points and 8 singularities.

- $\Delta \sim (u + \ldots)^4(u^4 + \ldots)$
- $\Delta \sim (u + \ldots)^3(u + \ldots)^2(u^3 + \ldots)$
- $\Delta \sim (u^3 + \ldots)^2(u^2 + \ldots)$

We need to carry out a systematic search for $\text{su}(2) \oplus \text{su}(2)$, $\text{su}(3)$, $G_2$, and $\text{sp}(2)$. 
Constructing New Seiberg-Witten Curves (sp(3) ⊕ su(2))

There are 6 singular points and 9 singularities.

- \( \Delta \sim (u + \ldots)^4(u^5 + \ldots) \)
- \( \Delta \sim (u + \ldots)^3(u + \ldots)^2(u^4 + \ldots) \)
- \( \Delta \sim (u^3 + \ldots)^2(u^3 + \ldots) \)

A systematic search reduces the problem to solving on the order of 800 polynomial relationships amongst 160 unknowns.
Constructing New Seiberg-Witten Curves (sp(5))

There are 7 singular points and 10 singularities

- $\Delta \sim (u + ...)^4(u^6 + ...)$

- $\Delta \sim (u + ...)^3(u + ...)^2(u^5 + ...)$

- $\Delta \sim (u^3 + ...)^2(u^4 + ...)$

A systematic search was not attempted for this case because of the outcome found on the previous slide.
IX Isogenies

An isogeny is a many-to-one holomorphic map that preserves the holomorphic 1-form. There are three traditional presentations of elliptic curves which are related by isogenies.

- Legendre: \( \eta^2 = \xi^3 + f\xi + g \)
- Jacobi: \( y^2 = x^4 + \alpha x^2 + \beta \)
- Hessian: \( \gamma = y^3 + \delta xy + x^3 \)

Where \( f, g, \alpha, \beta, \gamma, \) and \( \delta \) are all functions of \( u \).
Isogenies (2-isogeny)

The map from the Jacobi form to the Legendre form is a 2-isogeny.

- \( x = (\xi - \frac{1}{3} \alpha)^{\frac{1}{2}} \)

- \( y = \eta(\xi - \frac{1}{3} \alpha)^{-\frac{1}{2}} \)

- \( \Delta = \beta^2(\alpha^2 - 4\beta) \)

- The condition for a curve to have a 2-isogeny is that \( D(\beta) = kD(u) \) where \( k \in \mathbb{Z}^+ \).

The \( H_2, D_4, \) and \( E_7 \) curves have a 2-isogenous form. The \( H_2 \) isogenous curve can only have a \( u(1) \) flavor symmetry.
Isogenies (2-isogeny of $D_4$)

- $\alpha = \tau u + M_2$

- $\beta = u^2 + M_4$

- $\Delta = (u^2 + M_4)^2((\tau^2 - 4)u^2 + 2\tau M_2 u + (M_2^2 - 4M_4))$

- If we take the special case $M_4 = \frac{1}{4 - \tau^2} M_2^2$ then we get

\[
\Delta = (\tau^2 - 4)(u + \frac{\tau}{\tau^2 - 4} M_2)^2(u - (\tau^2 - 4)^{-\frac{1}{2}} M_2)^2(u + (\tau^2 - 4)^{-\frac{1}{2}} M_2)^2
\]

This is the same discriminant as the $N = 4$ solution.
Isogenies (2-isogeny of $E_7$)

- $\alpha = M_2u + M_6$

- $\beta = u^3 + M_8u + M_{12}$

- By comparing the dimensions of the complex parameters to the dimensions of the Casimirs of Lie Algebras the maximal flavor symmetry is $F_4$.

- A systematic computation of the SW 1-form still needs to be performed to see what are the possible flavor symmetries.
Isogenies (3-isogeny)

The map from the Hessian form to the Legendre form is a 3-isogeny.

- \( x = -(\xi - \frac{1}{12}\delta^2)(\eta + \frac{1}{2}(\delta\xi - \frac{1}{12}\delta^3 + \gamma))^{-\frac{1}{3}} \)

- \( y = (\eta + \frac{1}{2}(\delta\xi - \frac{1}{12}\delta^3 + \gamma))^{\frac{1}{3}} \)

- \( \Delta = \frac{1}{16}\gamma^3(\delta^3 - 27\gamma) \)

- The condition for a curve to have a 3-isogeny is the same as for a 2-isogeny \( D(\gamma) = kD(u) \) where \( k \in \mathbb{Z}^+ \).

The \( H_3 \) and \( E_6 \) curves have a 3-isogenous form. The \( H_3 \) isogenous curve can only have a \( u(1) \) flavor symmetry.
Isogenies (3-isogeny of $E_6$)

- $\delta = M_2$

- $\gamma = u^2 + M_6$

- By explicitly computing the Seiberg-Witten 1-form we find that the flavor symmetry of this curve is $G_2$.

- The discriminant has $Z = 4$ but it is hard to see how 6 neutral half-hypers could fit into a representation of $G_2$. 
Future Work

- Try to construct Seiberg-Witten curves for the $sp(3) \oplus su(2)$ and $sp(5)$ flavor symmetries. Systematic searches are plagued with technical difficulties.

- Carry out the remaining systematic searches for the rank 2 flavor symmetry solutions of the $E_6$ singularity.

- Try to determine the Seiberg-Witten 1-forms and flavor symmetries for the supposed $F_4$ mass deformation of the $E_7$ singularity.

- Try to better understand the relationship between isogenies and submaximal mass deformations.