Lecture II:
A. Entanglement in correlated systems
B. Introduction to topological defects

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Joel Moore
University of California, Berkeley
and Lawrence Berkeley National Laboratory

part B: Subroto Mukerjee (UCB), Cenke Xu (UCB/Harvard)
Ryan Barnett (Caltech)

Entanglement overview

0. What is entanglement entropy? Why compute it?

an "entangled" state $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$

1. What does it reveal about quantum critical states?

Quantum critical states have infinitely more entanglement than "typical" many-body states.

2. Applications:

(a) how does better understanding of entanglement entropy help with correlated quantum ground states? "*finite-entanglement scaling*"

(b) what happens to a quantum coherent system under a controlled nonequilibrium process (sweeping through a critical point)?

Entanglement helps us understand quantum criticality;
quantum criticality helps us do controlled nonequilibrium dynamics.

Quantum entanglement

Sometimes a pure quantum state of a bipartite system *AB* is also a pure state of each subsystem separately:

Example: S_z=1 state of two s=1/2 spins, A and B

$$|\Psi_{AB}\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle$$

a "product" state

Sometimes a pure quantum state of a bipartite system *AB* is **not** a pure state of each subsystem separately:

Example: singlet state of two s=1/2 spins

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an "entangled" state

"Maximal knowledge of the whole does not imply maximal knowledge of the parts"

Entanglement entropy

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an “entangled” state

In an entangled state, the state of subsystem A or B is not a pure quantum state, but rather a **density matrix**

For the singlet

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_B$$

A classical uncertainty or **entropy** has been created by the operation of looking at only part of the system.

Entanglement entropy

Definition: the entanglement entropy of a pure state,
with respect to a given partition into A and B,
is the von Neumann entropy of the partial density matrices

$$\langle\phi_1|\rho_A|\phi_2\rangle = \sum_j (\langle\phi_1| \times \langle\psi_j|)|\psi\rangle\langle\psi|(|\phi_2\rangle \times |\psi_j\rangle)$$

$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

In a diagonal basis, this is just $S = -\sum_i p_i \log_2 p_i$

The singlet generates one bit of classical entropy when the two spins are separated

Entanglement entropy

Thermalization hypothesis:
for a non-integrable quantum coherent system, the density matrix at long times converges *locally* to that of a thermal system.

$$\langle\phi_1|\rho_A|\phi_2\rangle = \sum_j (\langle\phi_1| \times \langle\psi_j|)|\psi\rangle\langle\psi|(|\phi_2\rangle \times |\psi_j\rangle)$$

$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

Note that the partial density matrix for subsystem A gives the results of *all* experiments limited to A

What we interpret locally as *thermal* entropy must come from *entanglement* entropy if the global system is phase-coherent.

How much entanglement entropy occurs in ground states of local Hamiltonians?

To get some intuition for how entanglement behaves in statistical physics, consider "valence bond states" of $s=1/2$ systems:

Rule: every spin forms a singlet with some other spin

Short-ranged VBS
(= "dimer covering")

Long-ranged VBS

In these states, entanglement entropy S just counts singlets:
S = 1 bit for each singlet crossing the AB boundary. (But real states are usually a bit more complicated.)

How much entanglement entropy occurs in ground states of local Hamiltonians?

Consider partitions of a *d*-dimensional infinite system *AB* into a subregion *A* of linear size *L* and an infinite subregion *B*.

How should entanglement entropy scale with *L*?

If we can ignore entanglement between points farther apart than some length scale ξ, then entanglement entropy should be determined by a shell of thickness ~ ξ around the *AB* boundary:

$$S \sim L^{d-1}\xi \Rightarrow S \sim L^{d-1} \text{as } L \to \infty \text{ with system parameters fixed}$$

the "area law"

If there is no notion of locality, any site in *A* is as likely to be entangled with a site in *B* as with another site in *A*, and $S \sim L^d$

What the area law tells us

In one dimension, the area law has been established for systems with an energy gap (for a review, see Eisert et al., arXiv 2008)

This area law also tells us that gapped systems in 1D are well approximated by "matrix product states" (defined momentarily) and can be simulated efficiently on a classical computer.

We can go beyond the area law

1. by looking at gapless states . Two examples follow.

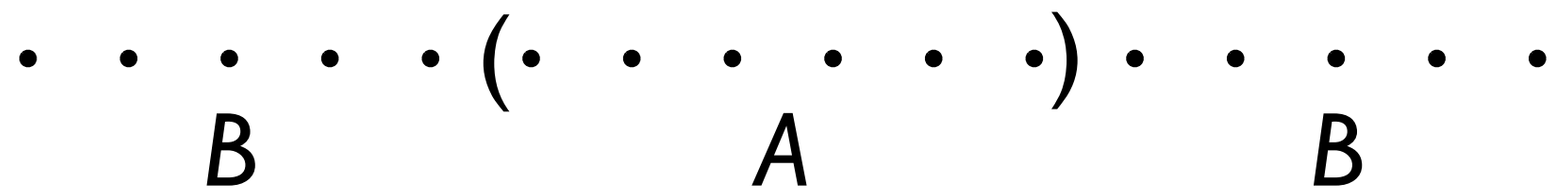
2. by looking at interesting subleading terms, e.g., topological entanglement entropy/spectrum

In higher dimensions, there can be area laws even in gapless systems, with interesting subleading parts (Ryu-Takayanagi, Fradkin-JEM, Metlitski et al.)

How much entanglement entropy occurs in ground states of local Hamiltonians?

We start with "pure" (translation-invariant), local Hamiltonians in one dimension.

Consider a partition for which *A* is a contiguous set of *N* spins inside an infinite chain:

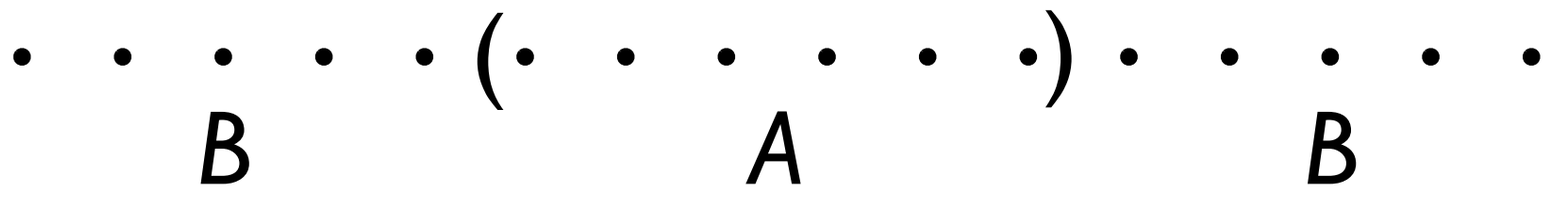


Away from critical points (i.e., when correlations are short-ranged), entanglement is localized in the vicinity of the boundary and the "area law" is satisfied:

$$\lim_{N\to\infty} S = C < \infty$$

But what about quantum critical states? Is there qualitatively more entanglement?

How much entanglement entropy occurs in critical states of local Hamiltonians?



Example of a quantum critical ground state: (c=1) Heisenberg AF

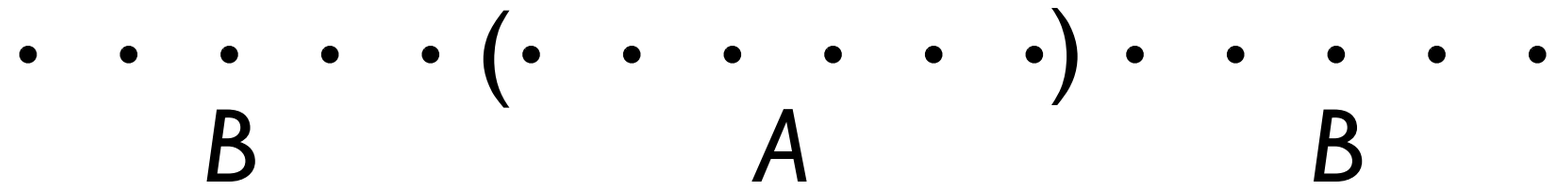
$$H = J \sum_i \mathbf{s}_i \cdot \mathbf{s}_j, \quad J > 0$$

At criticality, the entanglement of a connected subset of N spins, with the remaining spins, is (note: violates area law)

$$\lim_{N\to\infty} S \sim \frac{c}{3} \log N \to \infty$$

At clean and conformally invariant quantum critical points in d=1, there is logarithmically divergent entanglement with a coefficient related to the "central charge" of associated CFT. (Holzhey, Wilczek et al. 94, Vidal 03, Calabrese and Cardy 04).

Uses of entanglement entropy in d=1



For the subset of 1D quantum critical points that are described by 2D conformal field theories:

The appearance of the central charge in the ground-state entanglement is consistent with its appearance in other quantities related to entropy, such as the free energy at finite temperature

$$f = \frac{F}{L} = f_0 - \frac{\pi}{6} c (kT)^2 \hbar v$$

The central charge is an important quantity, but only defined for a subset of quantum critical points.

Entanglement entropy can be defined at *any* quantum critical point. Does it still show similar behavior, with a universal coefficient? *Yes!*

What about "applications"? Is this useful?

We want to apply knowledge about entanglement to improve our understanding of old-fashioned CM/AMO: correlations, phase diagrams, etc.

1. Quantum critical states have increased entanglement, sometimes with universal properties, both in 1D and higher dimensions.

2. It is believed that entanglement entropy underpins the best algorithms for correlated states and dynamics in 1D (and ground states in 2D) not amenable to quantum Monte Carlo.
Feiguin & White, Vidal, Verstraete & Cirac, Kollath, Schollwoeck, ...

Connection between (1) and (2): (1) Critical states of local Hamiltonians have "a moderate amount" of entanglement; (2) efficient numerical methods should use this property.

Studying quantum correlations with classical algorithms: applied entanglement entropy

Basic concept: "Entanglement entropy determines how much classical information is required to describe a quantum state."

Example:
how many classical real numbers are required to describe a *product* (not entangled) state of N spins?

simple product $|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$

Answer: ~ N (versus exponentially many for a general state)

How do we efficiently manipulate/represent moderately entangled states?

Applied entanglement entropy

The remarkable success of the density-matrix renormalization group algorithm in one dimension (White, 1992; Ostlund and Rommer, 1995) can be understood as follows:

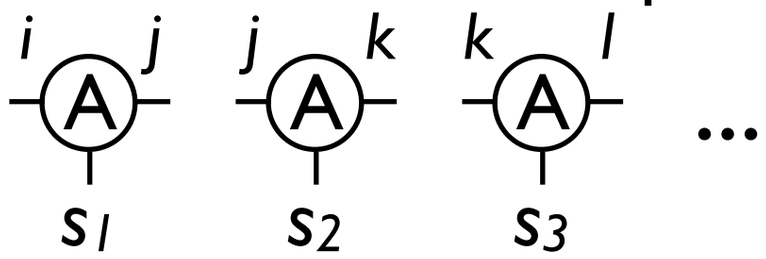
DMRG constructs "matrix product states" that retain local entanglement but throw away long-ranged entanglement.

Example states for four spins:

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Graphical tensor network representation:



Application I: finite-entanglement scaling

We want to understand how, if only finite entanglement can be retained (as on any computer), the physics of the state will be modified.

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Retaining finite entanglement is like a form of "image compression" on quantum states. How much information is lost depends on the complexity of the original state.

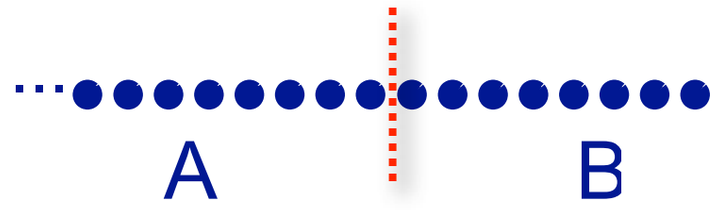
Quantum critical states were known (Tagliacozzo et al.) to develop an effective correlation length $$L_{\mathrm{eff}} \propto \chi^{\kappa}, \quad \chi = \dim A$$

What determines this "finite-entanglement scaling"?
Is it like "finite-size scaling"? (cf. Blöte, Cardy, & Nightingale)

Compute entanglement by writing a state in a simple basis:

- **Schmidt decomposition** of the state (**SVD**):
-

$$
\begin{aligned}
|\psi\rangle &= \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B \\
&= \sum_{\alpha=1}^{\min(N_A,N_B)} \lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B
\end{aligned}
$$



with $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$

- the natural measure of the entanglement is the **entropy:**

$$S_A = S_B = S = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

Practical representation of quantum states

- Hilbert-space dimension of many-body problems increases **exponentially** with number of sites
  example: spin 1/2 system on “classical” computers
  (store **one state** in double precision)

- need a useful way to “**compress**” quantum states so that the matrices studied remain finite-dimensional

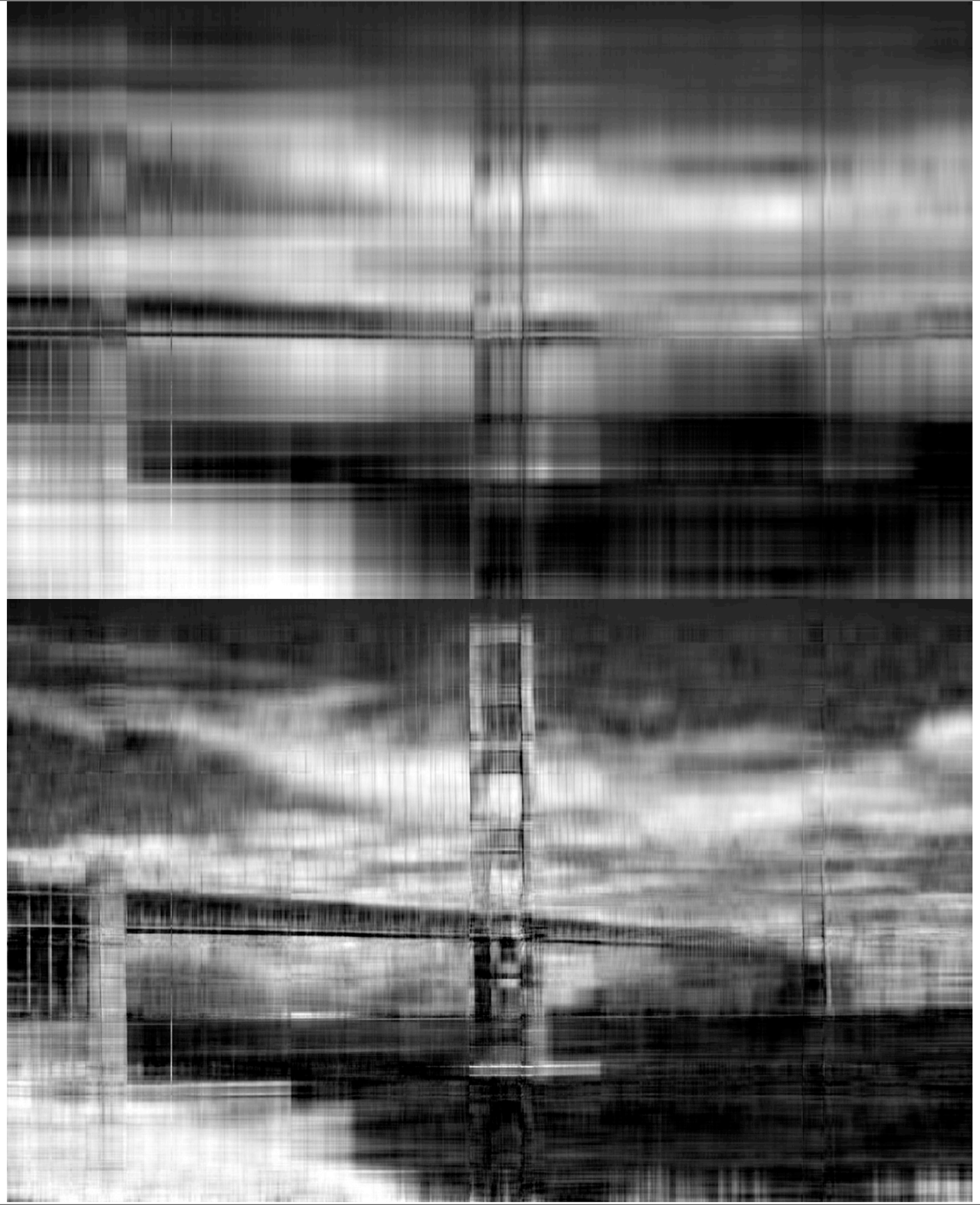
➡slightly entangled 1D systems: **Matrix Product States**

➡DMRG, TEBD, ...

➡New generation of “infinite system” (translationally invariant) algorithms: iTEBD, iDMRG

In practice, algorithms limit the dimension of the matrix product state to “compress” it: only the largest eigenvalues of the Schmidt decomposition are retained

$\chi = 4$

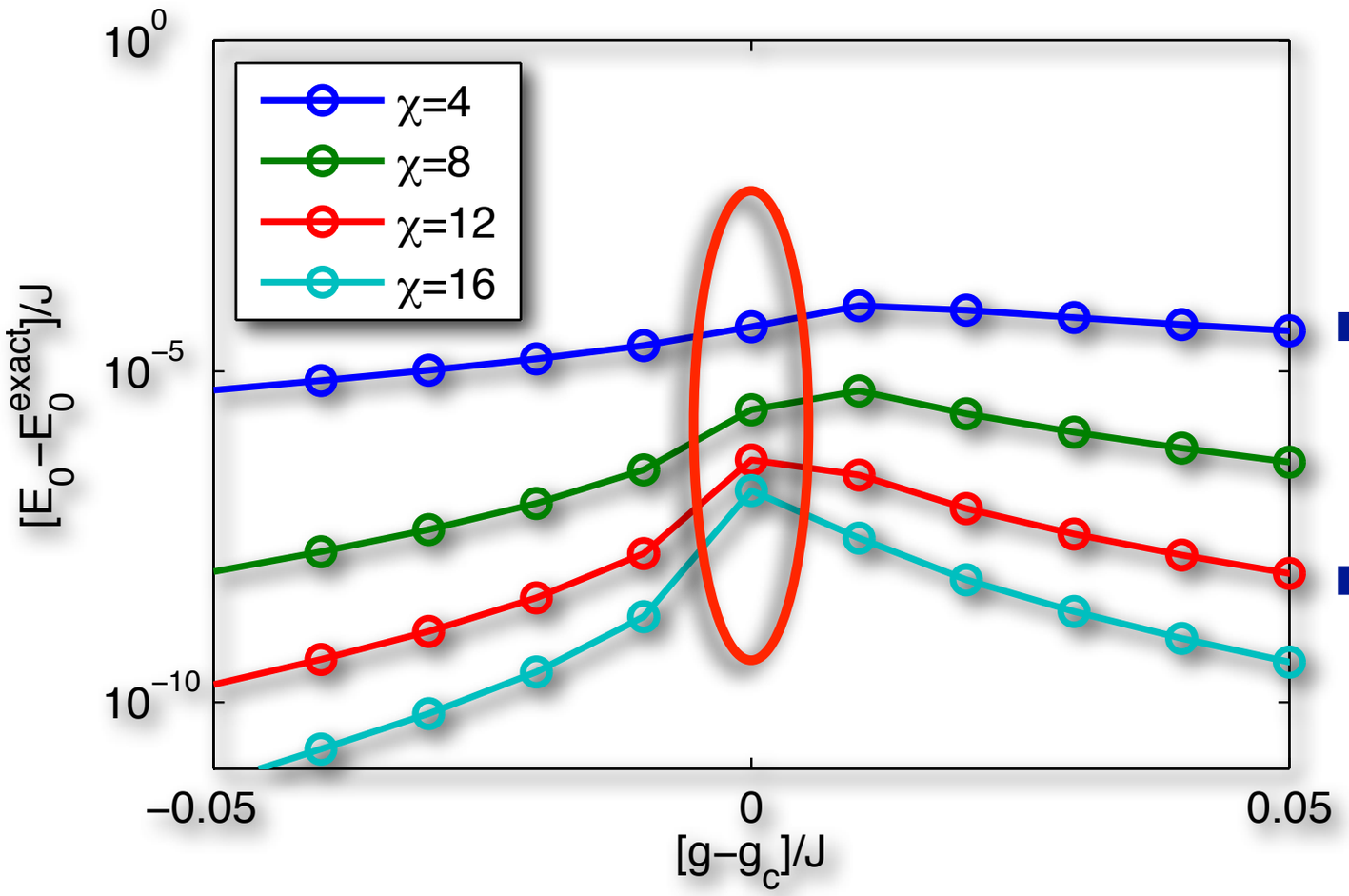


$\chi = 16$

$\chi = 64$

$\chi = 256$

- find the ground state of a system by using **imaginary time evolution** (almost unitary for small time steps)
- parallel updates for **infinite/translational invariant** systems: **iTEBD** [Vidal '07]
- **example**, transverse Ising model: $H = \sum_i \left(J\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x \right)$



➡convergence of wave function is worst at the **critical point**

➡conformal invariance

What goes wrong at quantum critical points?

We have a theory for critical points with **conformal invariance**: mapping on field theories in 1+1 dimensions

Basic idea: Consider minimizing energy within the matrix product states of a certain dimension *X*.

- This problem involves the sum of two energies: the real energy and a “truncation penalty”, the energy cost resulting from the truncation to finite *X*
- *For the above critical points, the knowledge of the “entanglement spectrum” of Schmidt eigenvalues (Calabrese and Lefevre) means that we can compute both terms in the effective energy, and minimize the sum.*

- (Li-Haldane) “entanglement spectrum” [Calabrese et al ‘08]

$$n(\lambda) = I_0\left(2\sqrt{-b^2 - 2b\log\lambda}\right)$$

of $\hat{\lambda}$ ‘s greater than λ

with $b = \dfrac{S}{2} = \dfrac{c}{12}\log\xi = -2\log\lambda_{\max}$

continuum of Schmidt values $|\psi\rangle = \sum_{\alpha=1}^{\infty}\lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B$

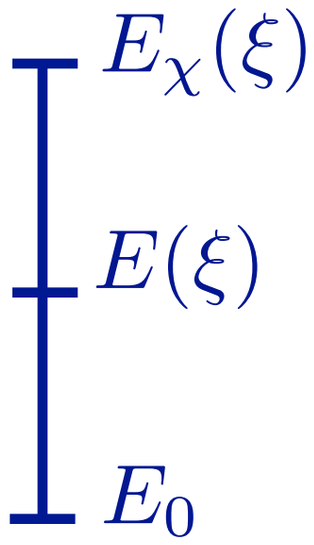
- **at a critical point**: all values are equal ($\lambda_\alpha \to 0$)

➡no meaningful way to truncate the sum at finite χ

Finite χ approximations at critical points will give **universal scaling forms**, e.g. $S = f(c)\log(\chi)$

- energy density of a truncated state

$$E_\chi(\xi) = E_0 + \frac{A}{\xi^2} + \frac{B}{\xi} P_r(\xi, \chi)$$



- $E_\chi(\xi)$ is a non-monotonic function

➡minimize the energy and find the **optimal correlation length for a fixed matrix dimension**

➡scaling relation $S = (c/6) \log \xi$ yields the entropy, etc.

- **we can find the best approximation of the critical state for a given number of states we keep**

- **analytical solution** for the asymptotic case
  (using a continuum of Schmidt values and $\chi \to \infty$)

  ➡**universal finite-entanglement scaling relations**

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}+1}\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}+1}} \log \chi$$

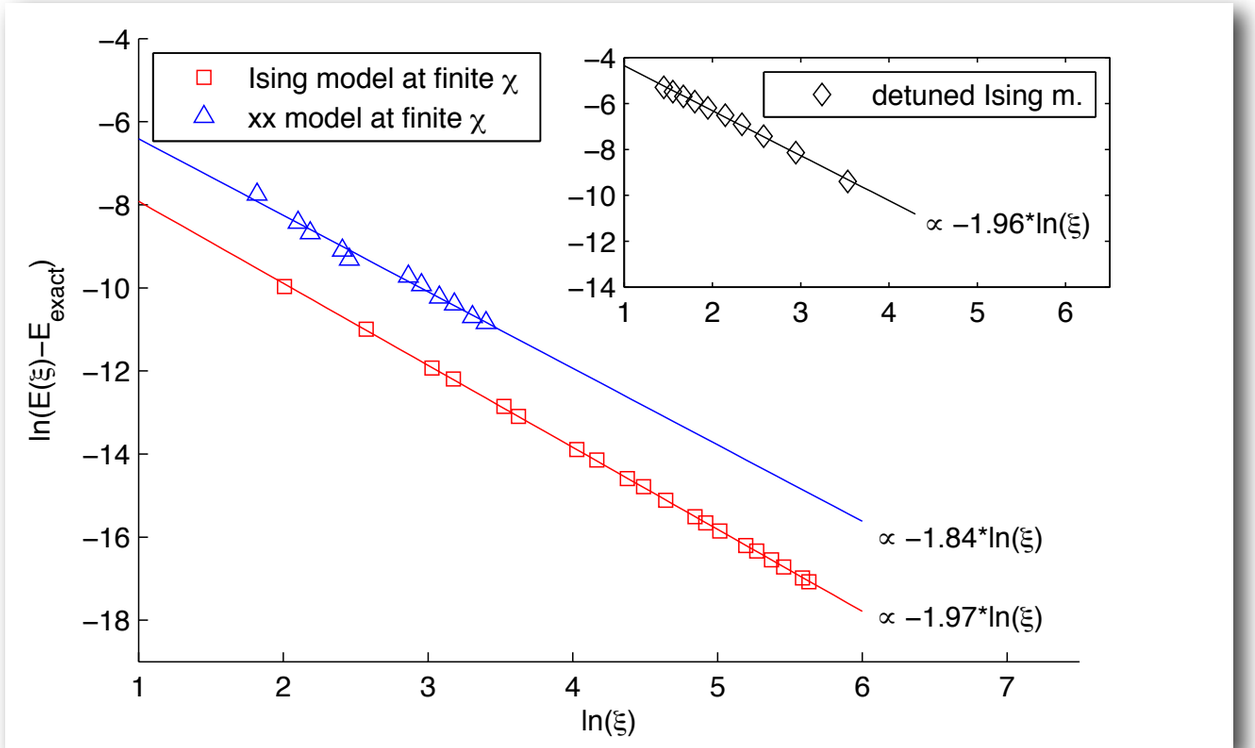
We believe this finite-entanglement scaling will result in any approach with finite matrix dimension, in the same way that finite-size scaling is "universal".

Now we can try to check this nonlinear *c* dependence:

(some more checks are in

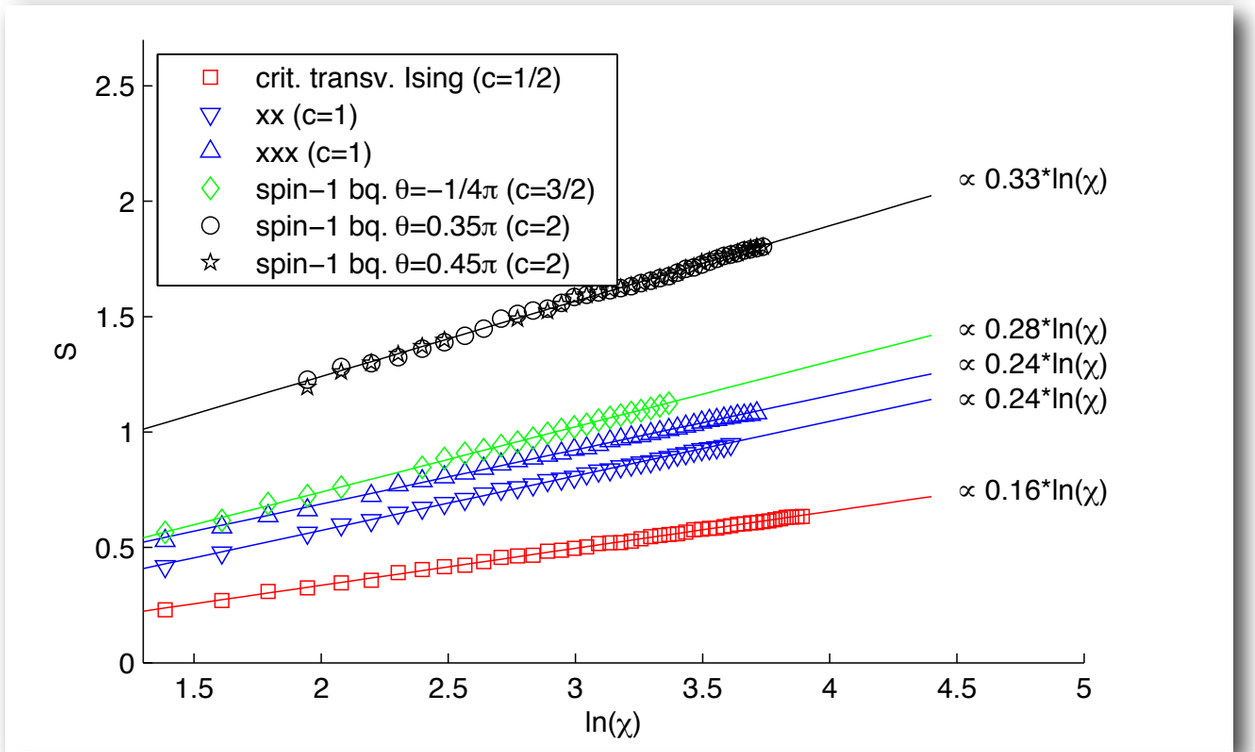
[F. Pollmann, S. Mukerjee, A. Turner, and J.E. Moore, PRL 2009])

- check scaling of the **energy** and **entanglement entropy:**



$$E_\chi(\xi) = E_0 + \frac{A}{\xi^2} + \frac{B}{\xi} P_r(\xi, \chi)$$

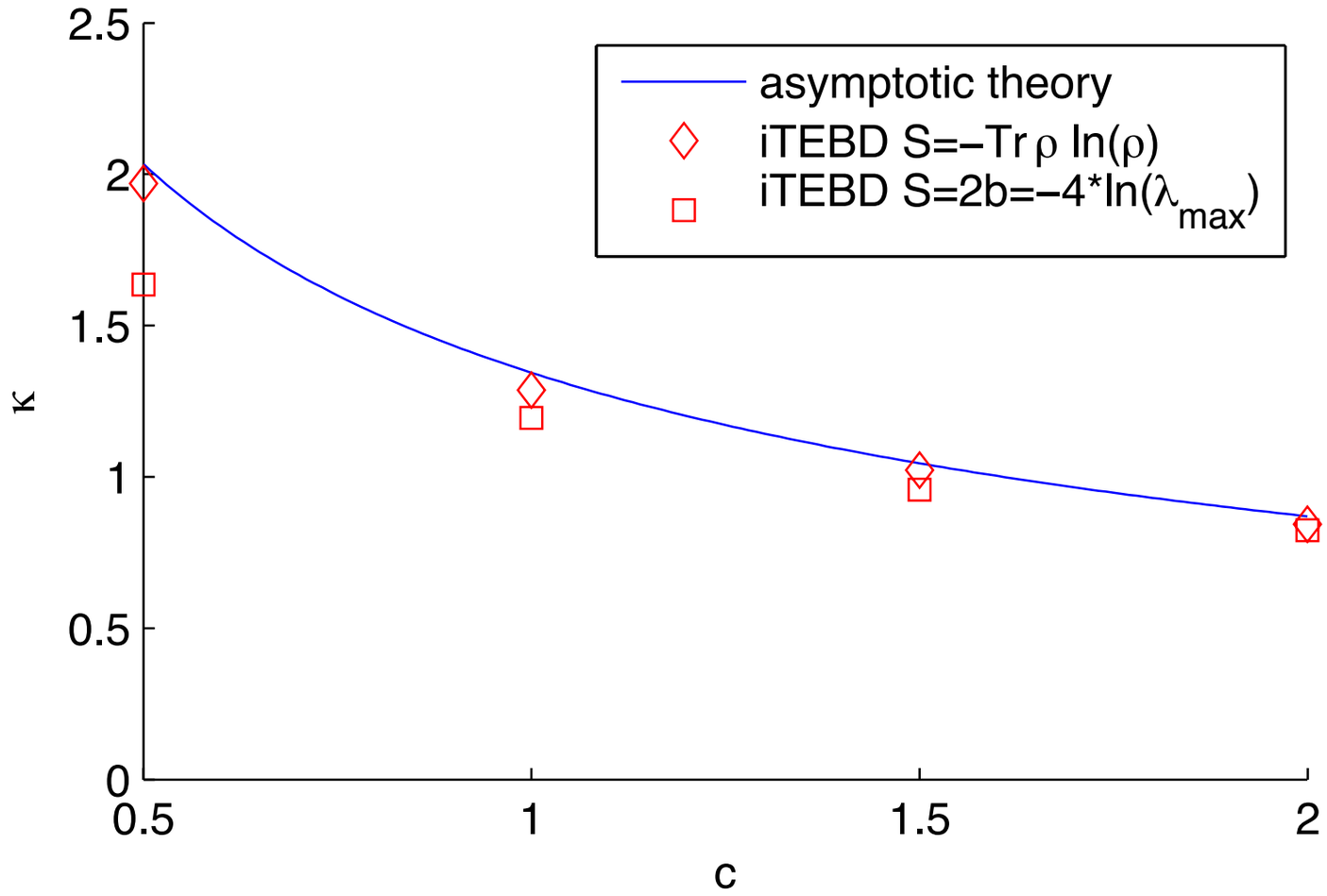
➡scales as $\propto 1/\xi^2$ for $\xi = \xi_{\text{opt}}$



$$S_\chi = \frac{1}{\sqrt{\frac{12}{c} + 1}} \log \chi$$

➡$S_\chi \propto \log \chi$ and depends only on c

- reasonable agreement of the **asymptotic theory** and **numerical results**



- Errors are no larger than differences between different definitions of entropy
- Another check: combine non-interacting copies; still get nonlinear dependence on total *c*

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}}+1}\log\chi$$

- new asymptotic scaling law for the finite-entanglement scaling of 1D quantum-critical systems

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}+1}\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}+1}} \log \chi$$

➡reasonable agreement with numerical results for accessible χ

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➡reasonable agreement with numerical results for accessible χ

- future directions:

➡critical points with disorder (known entanglement spectrum)

➡universal scaling in dynamical properties of QCP (next)

IIB: "Topological defects", using 2D spinor condensates as an example

Why spinor condensates?

1. Possibility of interesting magnetic ordering: Broken symmetry more than just U(1)

2. Interesting phase transitions and topological defects

3. Realized experimentally in the last 10 years

Why 2D?

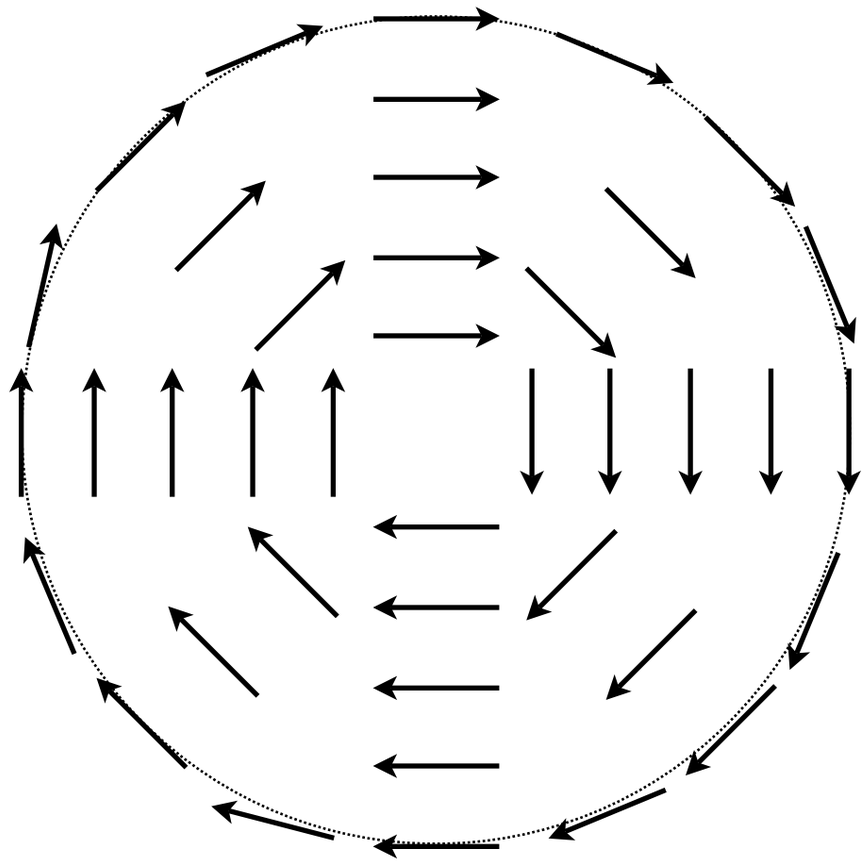
1. Unconventional phases and phase transitions

2. Topology and defects play an important role

3. Realized experimentally in the last few years

Vortices in two dimensions

A vortex in a superfluid is a special type of excited state. Far away from its "core", the superfluid is locally in a ground state, but the phase rotates smoothly as we move around the core.



If we circle the core at long distance in real space, the local phase winds in the "order parameter manifold" (in this case, the circle of possible phases).

Notes:
1. There is no well-defined phase at the core; there, the system is not locally in a ground state.
2. Vortices are related to maps from one circle (around the core in real space) to another.

$$e^{i\phi} = s_x + i s_y$$

Vortices can be important dynamically (they are typically long-lived) but also statically. When are vortices macroscopically significant?

General method for "topological defects", including vortices

Suppose we have a transition between a disordered phase ("high temperature") and an ordered one ("low temperature").

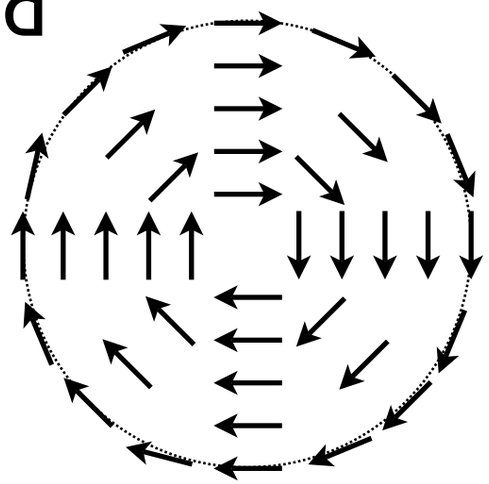
A. Figure out the symmetry groups G (disordered) and H (ordered).

B. Since a symmetry was broken, H is smaller than G.
We label "equivalence classes of ordered states" by the manifold M = G/H.
This means we look at elements of G and identify two differing by an operation of H.
The "order parameter manifold" is equal to G if H=1, i.e., symmetry is completely broken.
A somewhat nontrivial example is a collinear magnet: G = SO(3), H = SO(2), M = S^2
(surface of the unit sphere).

C. Point defects in 2D are labeled by mappings from the circle to M;
Line defects in 3D are also labeled by mappings from the circle to M;
Point defects in 3D are labeled by mappings from the sphere to M;
and so on.

Generalized vortices: "topological defects"

- 2D point or 3D line topological defects are equivalent to non-contractable closed loops on order parameter manifold



$$f : S^1 \rightarrow S_1, \quad f(\theta) = \theta$$

$$n\text{-vortex} : f(\theta) = n\theta, n \in \mathbb{Z}$$

- Defects form a group called the first homotopy or fundamental group of the manifold (denoted as $\pi_1(M)$)
- Concatenation of defects is the group multiplication operation
- Fundamental group need not be Abelian

Two macroscopic effects from vortices

In a rotated superfluid, a triangular “vortex lattice” forms; this is the minimum-energy configuration for the logarithmic interaction between vortices.

(Actually this lattice is most easily observed in type-II superconductors.)

Two macroscopic effects from vortices

In a rotated superfluid, a triangular “vortex lattice” forms; this is the minimum-energy configuration for the logarithmic interaction between vortices.

(Actually this lattice is most easily observed in type-II superconductors.)

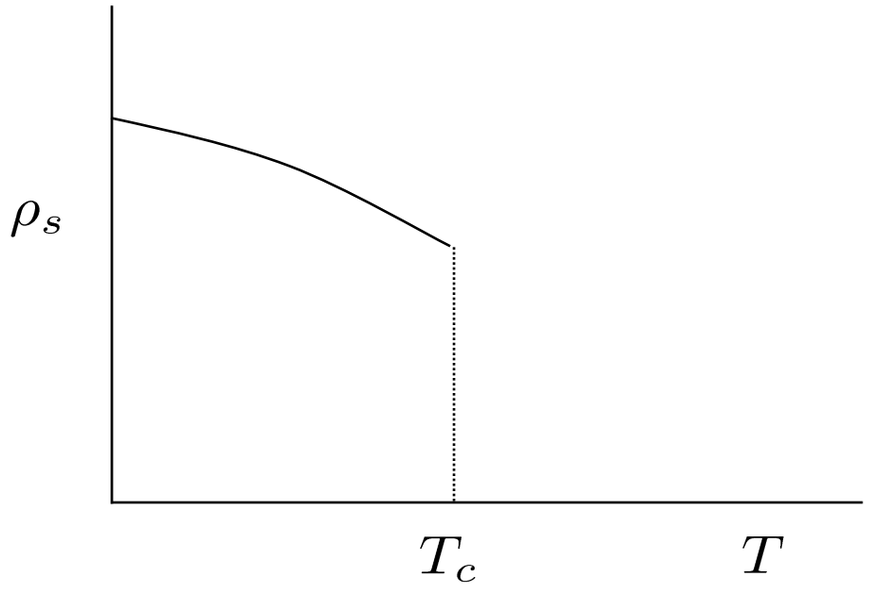
A more subtle effect is the vortex-unbinding (BKT) transition in a 2D superfluid.

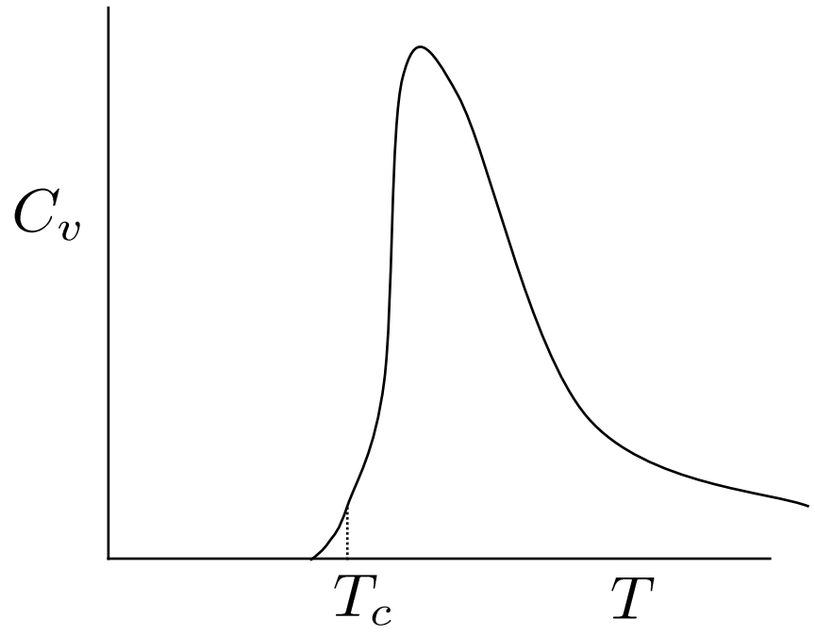
In two dimensions (i.e., superfluid films), the existence of superfluidity is somewhat surprising: there is no noninteracting Bose-Einstein condensation (no macroscopic occupancy of a single state), and a theorem forbids true long-range order.

Instead there is power-law order $\langle \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(0) \rangle \sim r^{-\eta}$

and a transition which can be viewed as unbinding of vortices: the system becomes disordered when the stiffness is too low to bind vortices into vortex-antivortex pairs. This leads to a universal superfluid stiffness jump at the transition.

2D superfluid (BKT) transition for spinless bosons

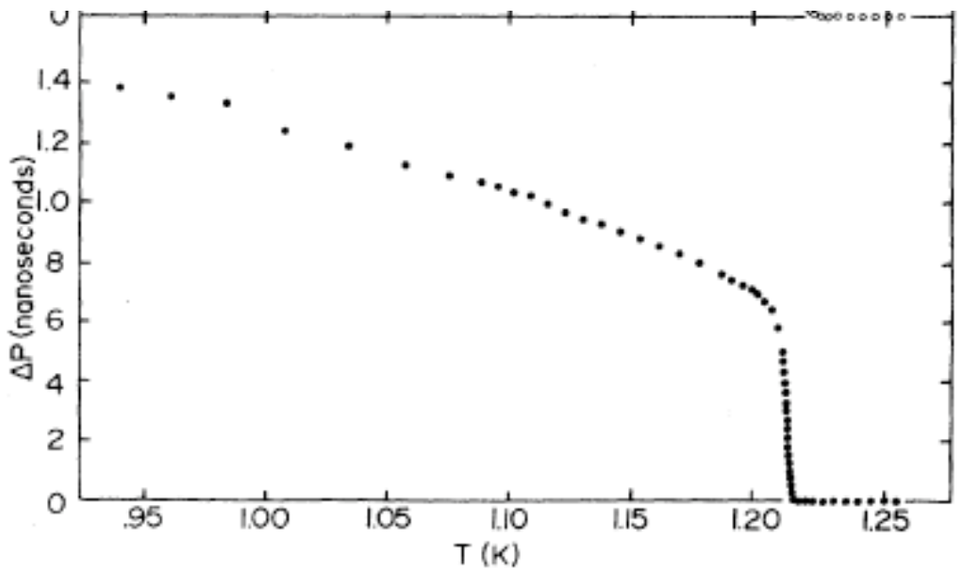




$$\frac{\rho_s(T_c^-)}{T_c} = \frac{2m^2}{\pi\hbar^2 k_B}$$

Universal jump

C_v has weak essential singularity at T_c



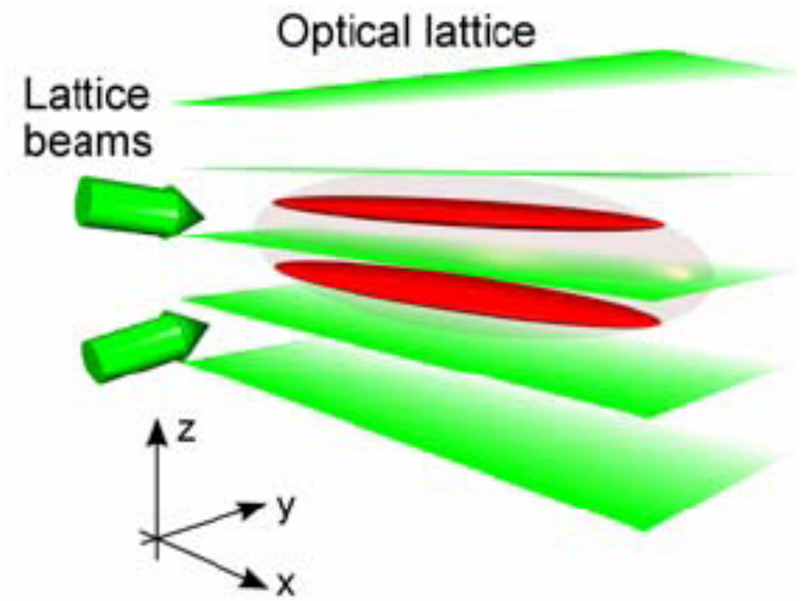
Experiment: Bishop and Reppy
PRL 40, 1727 (1978)

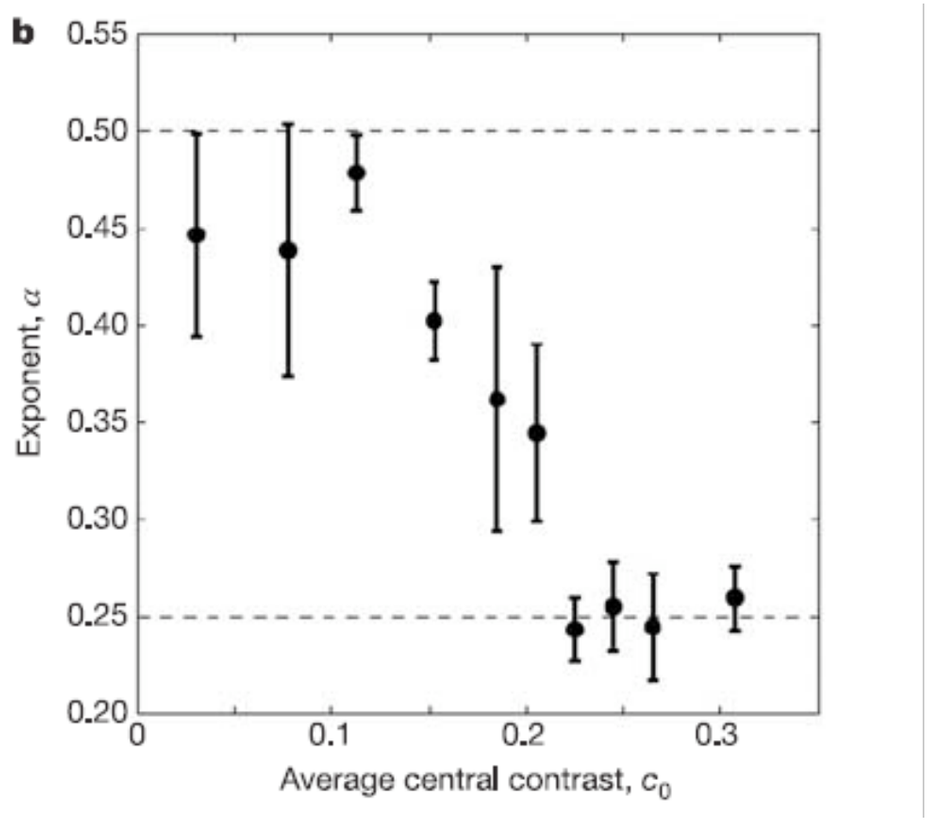
Thin helium film: torsional oscillator

KT transition in atomic BEC in a magnetic trap (^{87}Rb)

Hadzibabic et. al, Nature 441, 1118 (2006)

Interference measurement





c_0 like temperature. High c_0, low temperature.

$$\left\langle C^2(L_x)\right\rangle \approx \frac{1}{L_x}\int_0^{L_x} dx\,\left[g_1(x,0)\right]^2 \propto \left(\frac{1}{L_x}\right)^{2\alpha}$$

$\alpha = 0.5$, short ranged correlations

Stiffness jump corresponds to $\alpha = 0.25 \rightarrow 0.5$

Spinor bosons in the continuum

Starting model will be s=1 bosons with local spin-dependent interaction (to be justified in a moment)

$$H = \int d\mathbf{r}\Big(\frac{\hbar^2}{2M}\nabla\psi_a^+\cdot\nabla\psi_a + U(\mathbf{r})\psi_a^+\psi_a + \frac{c_0}{2}\psi_a^+\psi_b^+\psi_b\psi_a + \frac{c_2}{2}\psi_a^+\psi_{a'}^+\mathbf{F}_{ab}\cdot\mathbf{F}_{a'b'}\psi_{b'}\psi_b\Big)$$

Goal for part I:
show that even the simplest phases of nonrotated s=1 bosons have nontrivial topological defects and associated phase transitions

Recall that the thermal transition out of an ordinary (spinless) 2D superfluid is caused by *vortex unbinding*:

the interaction between vortices is logarithmic in separation L
the positional entropy per vortex (~ log A) is logarithmic in separation
=> there is a transition at a certain ratio of kT to stiffness

Effective two-body interaction in an optical trap

$$V_{12} = \delta(|\mathbf{r}_1 - \mathbf{r}_2|) f(\vec{S}_1, \vec{S}_2)$$

T. –L. Ho, PRL 81, 742 (1998)
A. J. Leggett, RMP 73, 307 (2001)

Only s wave scattering: $k_B T / E_c << 1$
Diluteness

E_c: Energy of bound state

$$f(\vec{S}_1, \vec{S}_2) = g_0 P_0 + g_2 P_2$$

for *S*=1 atoms

Only even spin projections allowed

$g_0 > 0, g_2 > 0$ for ^{23}Na and ^{87}Rb

$$V_{12} = \delta(|\mathbf{r}_1 - \mathbf{r}_2|)(g_0 P_0 + g_2 P_2)$$
$$= \delta(|\mathbf{r}_1 - \mathbf{r}_2|)(c_0 + c_2 \vec{S}_1 . \vec{S}_2)$$

$c_0 = \frac{g_0 + 2g_2}{3}$ $\quad c_2 > 0$ Antiferromagnetic or Polar (^{23}Na)

$c_2 = \frac{g_2 - g_0}{3}$ $\quad c_2 < 0$ Ferromagnetic (^{87}Rb)

s=1 bosons

$$H = \int d\mathbf{r}\Big(\frac{\hbar^2}{2M}\nabla\psi_a^+\cdot\nabla\psi_a + U(\mathbf{r})\psi_a^+\psi_a + \frac{c_0}{2}\psi_a^+\psi_b^+\psi_b\psi_a + \frac{c_2}{2}\psi_a^+\psi_{a'}^+\mathbf{F}_{ab}\cdot\mathbf{F}_{a'b'}\psi_{b'}\psi_b\Big)$$

$$\psi = \sqrt{n_0}\zeta$$

Two simple T=0 mean-field phases (Ho, 1998)

c2>0 polar

c2<0 ferromagnetic

$$\zeta_P = e^{i\theta}U\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \zeta_F = e^{i\theta}U\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Here theta is a phase and U is a spatial rotation matrix (SO(3)).

s=1 bosons

What are the order parameter manifolds in these phases?

Need to find cosets

$$M = \frac{G}{H}, \quad G = U(1)_G \times SO(3)_R, \quad H = \text{residual symmetry}$$

Results: $M_F = SO(3)$ (which also appears in 3He)

$$M_P = \frac{S^1 \times S^2}{\mathbb{Z}_2} \qquad \mathbb{Z}_2 = \text{identify } (\theta, \hat{\mathbf{n}}) \leftrightarrow (\theta + \pi, -\hat{\mathbf{n}})$$

$$\zeta_P = e^{i\theta} \begin{pmatrix} \frac{e^{i\alpha}}{\sqrt{2}} \sin\beta \\ \cos\beta \\ \frac{e^{-i\alpha}}{\sqrt{2}} \sin\beta \end{pmatrix}$$

Summary of topological defects in s=1 phases

Ferromagnetic condensate has Z_2 defects

No point defects exist in 3D ($\pi_2(SO(3)) = 0$)

Polar condensate has vortices

Fundamental vortex combines π phase rotation and magnetic inversion

3D point defects (monopoles) exist: $\pi_2\left(\frac{U(1)\times S^2}{Z_2}\right) = Z$

Review of 2D finite temperature behavior of continuous order parameters

- Mermin-Wagner: No long-range order

- Quasi-long-range order possible (as for spinless bosons)
- Low T behavior describable in terms of Non-linear Sigma Model

- Perturbative Renormalization Group (RG) flows can be calculated for the stiffness of the order parameter.
- Perturbative RG flows depend only on the local structure of the order parameter manifold

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- Perturbative RG flows depend only on the local structure of the order parameter manifold

$$\frac{dT}{dl} = -(d-2)T + (n-2)K_d \Lambda^{d-2} T^2 + \dots$$

In 2D for $n > 2$ flow is away from $T = 0$
No order at finite T

$n = 2$ ($U(1)$), marginal to all orders in 2D

Line of critical points

s=1 bosons

Concentrate on the polar state in 2D:

Locally the manifold looks like a spin part on the sphere (which flows to high temperature), and a phase part with no flow in perturbation theory.

Thus at small finite temperature the system is in a "nematic superfluid" phase: the spin part is disordered, and the phase part is only defined modulo π.

$$M_P = \frac{S^1 \times S^2}{\mathbb{Z}_2} \qquad \mathbb{Z}_2 = \text{identify } (\theta, \hat{\mathbf{n}}) \leftrightarrow (\theta + \pi, -\hat{\mathbf{n}})$$

This phase supports **half-vortices** (vortices with half the total boson circulation of a vortex in a single-component superfluid).

It has power-law correlations

$$\left\langle e^{2i\theta(0)} e^{-2i\theta(r)} \right\rangle \sim \frac{1}{|r|^{\Delta}}$$

2D Polar condensate summary

Fundamental vortex has 1/2 flux quantum

KT transition mediated by half vortices

Stiffness jump 4 times regular KT jump

Algebraic order in 2θ instead of θ (2D nematic)

Order parameter: $\psi_0^2 - 2\psi_{+1}\psi_{-1}$

Low T phase: paired singlets of bosons

Pairing induced by **thermal fluctuations**

Check with Monte-Carlo simulations

Free energy functional

$$F = \int d\mathbf{r}(\frac{\hbar^2}{2M}\nabla\psi_a^*\cdot\nabla\psi_a + a(T)\psi_a^*\psi_a + b_0\psi_a^*\psi_b^*\psi_b\psi_a + b_2\psi_a^*\psi_{a'}^*\vec{S}_{ab}\cdot\vec{S}_{a'b'}\psi_{b'}\psi_b)$$

Phase stiffness

$$\Gamma = \left(\left\langle\frac{\partial^2 F}{\partial\phi^2}\right\rangle - \frac{1}{k_B T}\left\langle\left(\frac{\partial F}{\partial\phi}\right)^2\right\rangle\right)\Bigg|_{\phi=0}$$

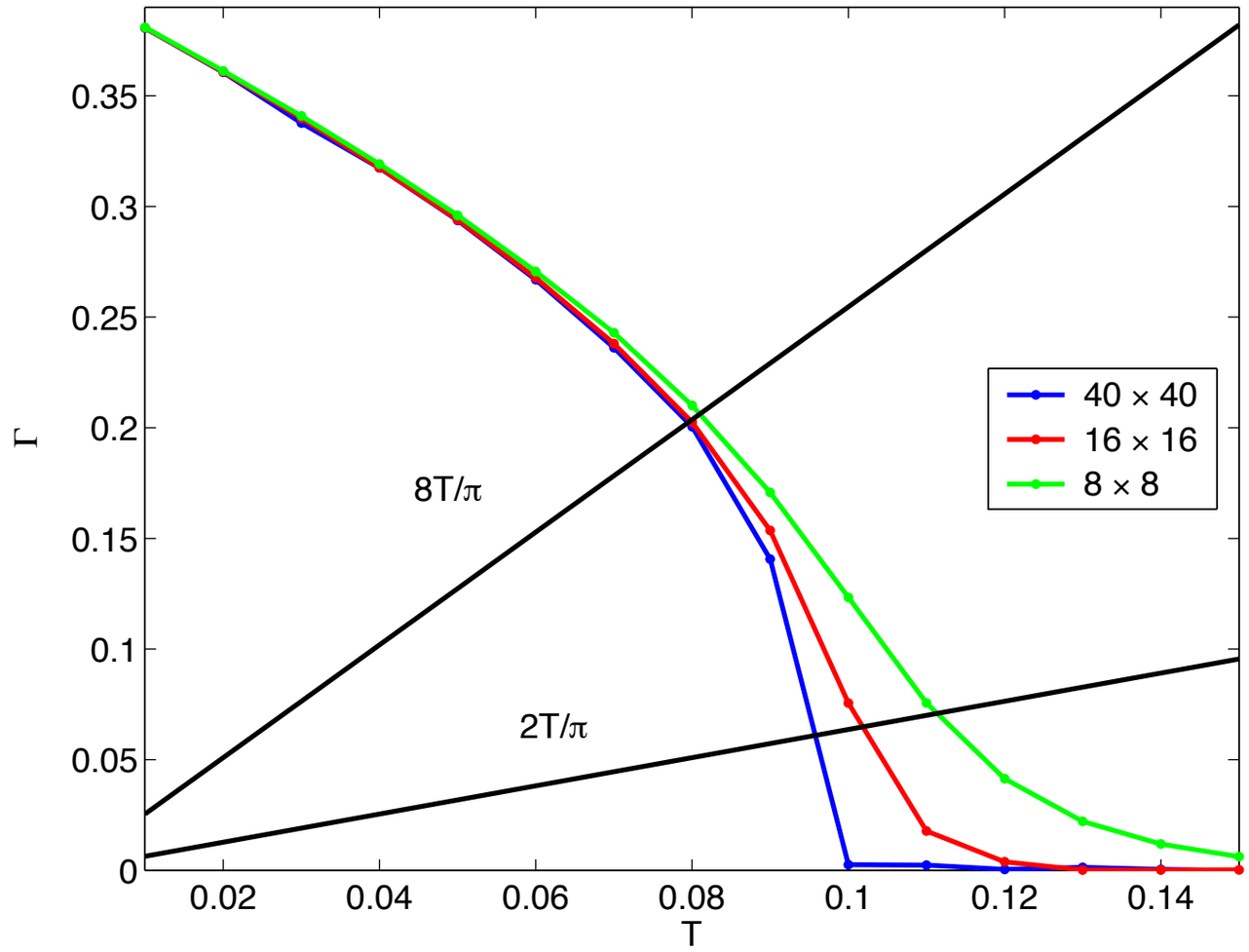
ϕ is a twist in θ

Smoking gun for numerics (and possibly experiments)

The transition out of this phase should be a Kosterlitz-Thouless transition driven by unbinding of half-vortices.

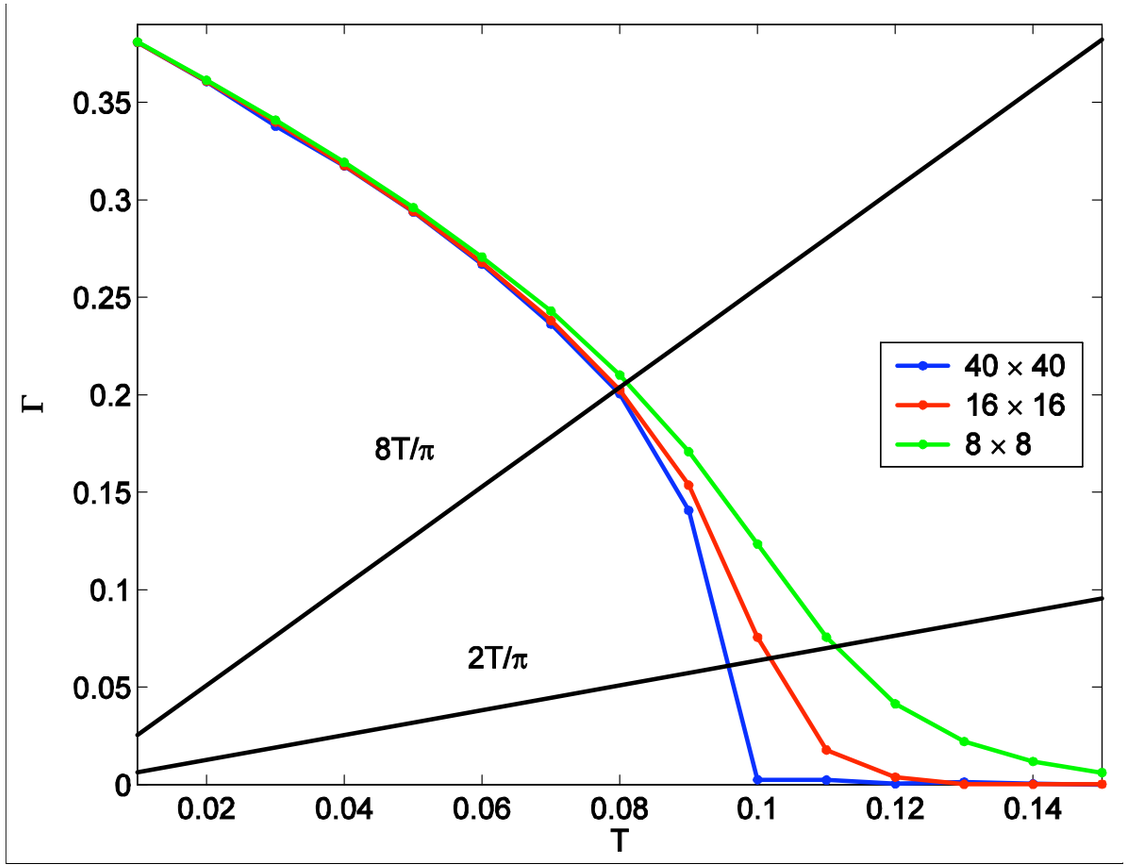
If so, there is a universal stiffness jump four times the usual value:

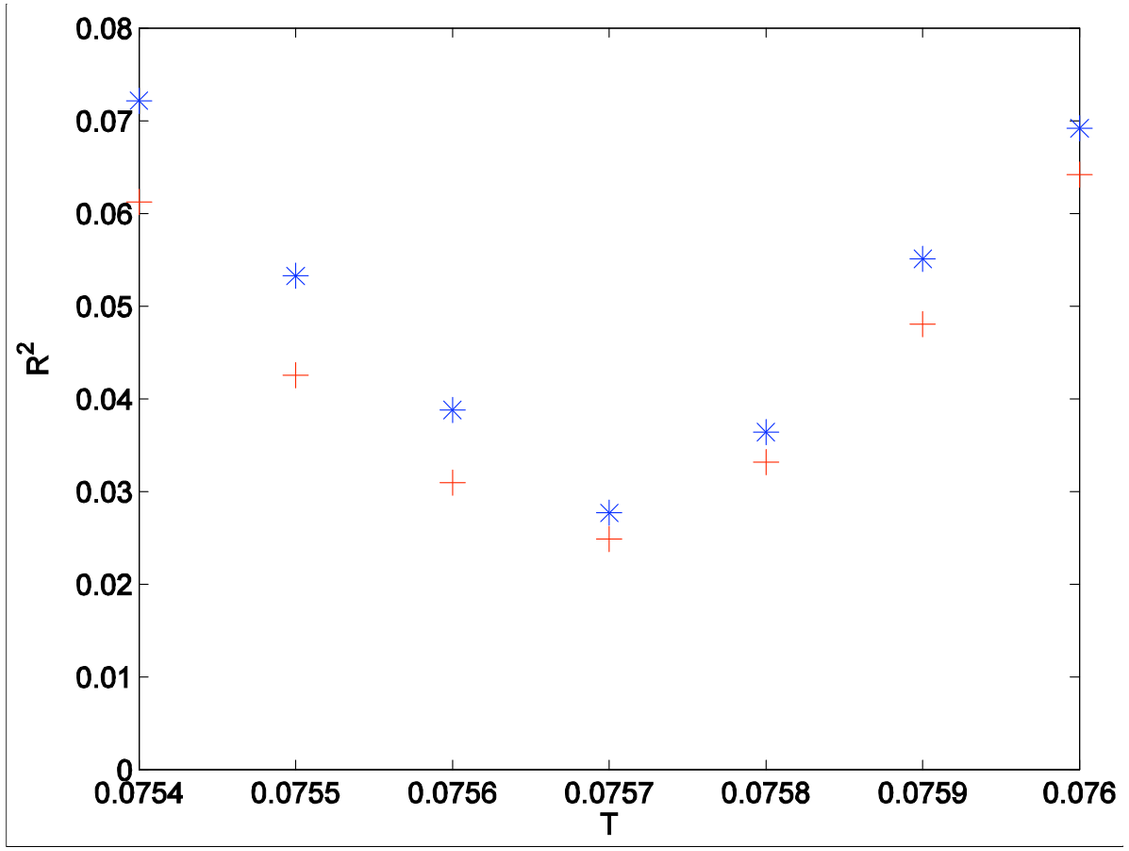
$$\rho_\infty(T_c) = \frac{8T_c}{\pi}$$



Mukerjee, Xu and Moore, Phys. Rev. Lett., 97, 120406 (2006) and cond-mat/0605102

2D finite temperature





$$\Gamma(N, T_c) = \Gamma_\infty \left[1 + \frac{1}{2} \frac{1}{\ln N + C} \right]$$

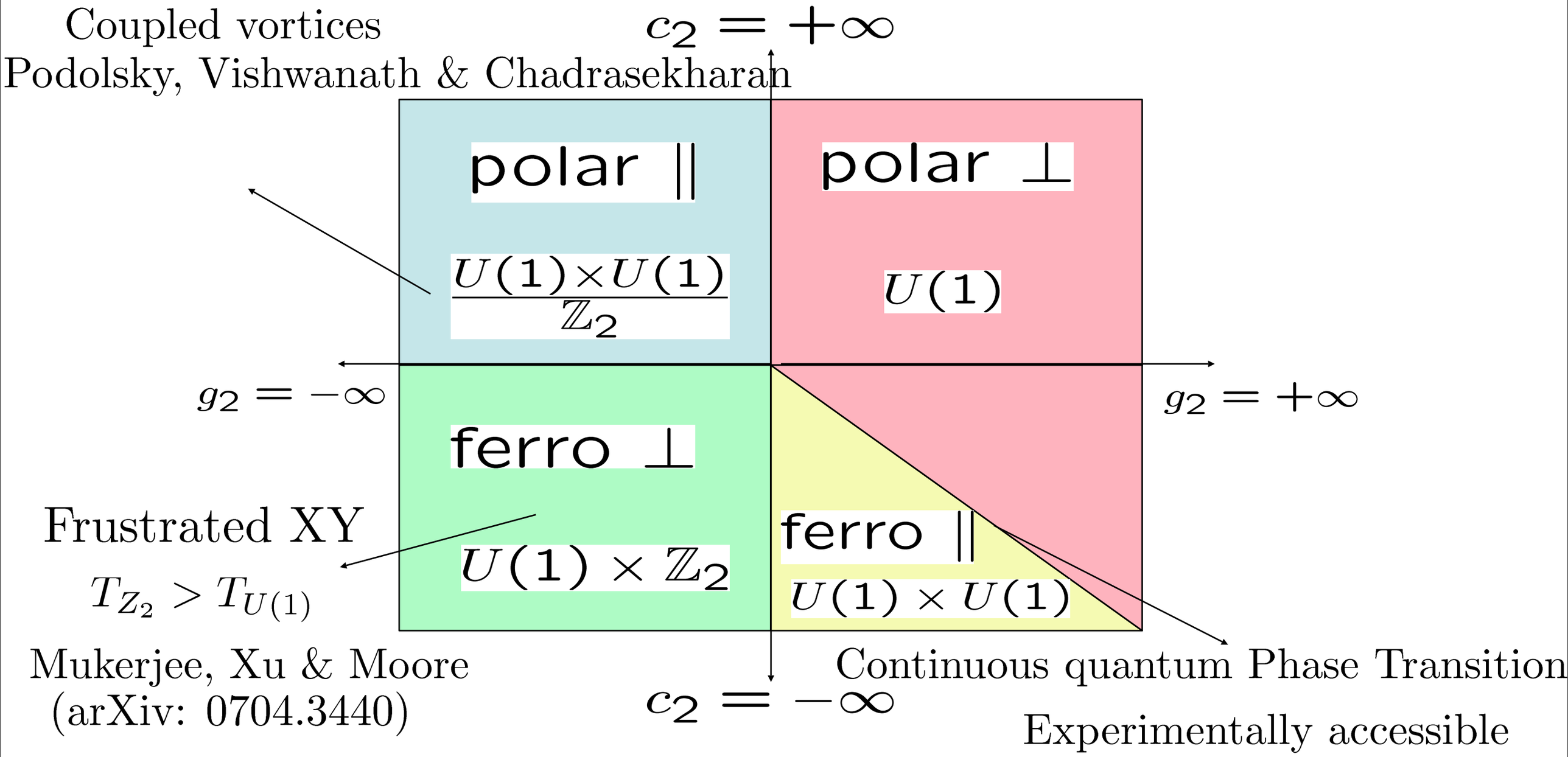
$$\Gamma_\infty = 8 T_c / \pi$$

Also in two-color lattice QCD in 2+1 dimensions
Chandrasekharan, PRL 97, 182001 (2006)

Can the same transition be driven by quantum fluctuations at T=0
in d=1 s=1 Bose-Hubbard model at commensurate filling?

Experiments often have quadratic Zeeman

$$E = n^2(c_0 + c_2\langle\vec{S}\rangle^2 + g_2\langle S_z^2\rangle)$$



A physical consequence of defects in 3D: vortex lattice matter

Consider an ordinary (spinless) superfluid in 3D.

The irrotational nature of the superflow means that rotation produces defects (vortices). The ground-state configuration is a triangular lattice of fundamental vortices. (Tkachenko)

The vortex density is determined by the rate of rotation.

Fundamental vortices appear because the interaction between vortices, at fixed angular momentum, is a logarithmic repulsion: R=system size, L=separation, a0=core

$$\Delta E \approx (K_c/2)\log(R/L) - (K_c/2)\log(R/a_0) = -(K_c/2)\log(L/a_0)$$

Spinor condensates

Spinor condensates have more complicated defect lattices for two obvious reasons: there are multiple types of defects, and multiple stiffnesses (e.g., spin and charge for s=1 polar).

A less obvious reason: non-additive forces for Abelian but non-integral defects.

Consider an explicit example: s=1 polar isotropic. Defects have an integer or half-integer "phase" winding, and a Z2 "spin" part determined by the phase.

$$A = \{n, e\} \qquad B = \{n + 1/2, g\}$$

Quadratic Zeeman rescues us

The good news is that for most current experimental systems, quadratic Zeeman reduces the spin symmetry to U(1) and all defect charges and forces are additive.

For s=1 polar, need to consider (1,0) and (1/2,±1/2).

The energy calculation of which lattice is favored reduces essentially to a two-component problem solved by Mueller and Ho. Numerical result is
Ks>Kc: triangular lattice of (1,0) vortices
Kc<Ks: first, honeycomb lattice of (1/2,±1/2) vortices, then jumps and deforms to square lattice, then ???.
Spin interactions favor *bipartite* lattices; spin and charge interactions frustrate each other.

The calculation can be simplified considerably by Ewald summation rather than velocity field integration....

Ewald sum for spinor vortex lattices

Need to compute potential for one vortex at the origin from all other vortices. Trick: add and subtract Gaussian screening potentials,

$$q(r) = \delta(r) - \frac{1}{\pi\sigma^2} e^{-r^2/\sigma^2}$$

then use Poisson resummation for Gaussian parts. Result

$$\phi(0) = \frac{1}{2} \sum_{\mathbf{R}\neq 0} \mathrm{Ei}\left(\frac{R^2}{\sigma^2}\right) + \sum_{\mathbf{G}\neq 0} \rho_0 \frac{2\pi}{G^2} e^{-\sigma^2 G^2/4} - \log\left(\frac{\xi}{\sigma}\right) - \frac{\gamma}{2} - \rho_0 \frac{\pi}{2}\sigma^2$$

converges within 5 terms or so.

Check: energy per vortex for square and triangular lattices.

$$E_{\mathrm{triangular}} = (K/2)(-\log(a_0\sqrt{n}) - 1.3211)$$

$$E_{\mathrm{square}} = (K/2)(-\log(a_0\sqrt{n}) - 1.3105)$$

General approach to spinor vortex lattices

Simple to generalize to multiple additive charges. We have computed spin-2 lattices. Special features of spin-2:

Three phases: ferromagnetic, "cyclic", polar

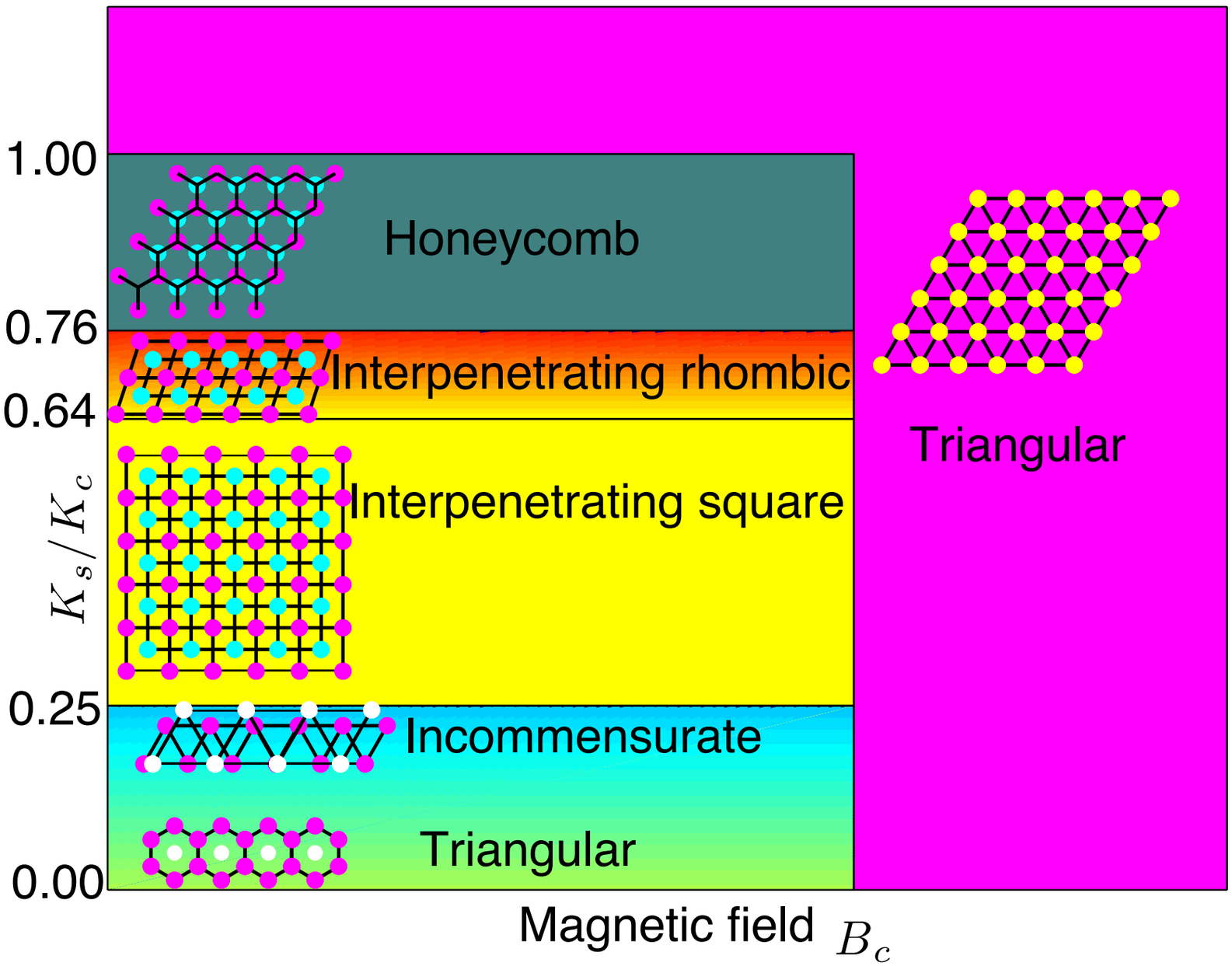
The cyclic or "tetrahedral" phase has non-Abelian fundamental group if isotropic. If anisotropic, defects are Abelian and in one configuration have 1/3 charges.

Must the ground state always be a true lattice? (no liquid/quasicrystalline state?) No! (aperiodic vortex structures)

What are the Tkachenko modes? Are they observable?

What about competing states at low temperature/high rotational velocity (e.g., QHE states of spinor bosons)?

Theoretical cyclic phase diagram



(Note: Ks/Kc is related in a complicated way to temperature and spin anisotropy)

Conclusions/the future

Conclusion: adding spin to bosons modifies the BKT transition, because of fluctuation-induced pairing, and the vortex lattice, because of complex intervortex interactions.

Topological defects are *especially robust excited states* of systems with conventional "symmetry breaking" order.

Are there ordered systems where topology appears even in the definition of the order?

Yes: topological phases! First 3D topological phase observed in experiment: "topological insulator" (2008)