BOUNDS ON MODULI MASSES IN CALABI-YAU STRING MODELS

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- Constraints on SUSY breaking in SUGRA
- Metastability condition and mass bounds
- Application to constant curvature spaces
- Implications for moduli in string models
- Detailed analysis for Calabi-Yau models

Based on works with L. Covi, M. Gomez-Reino, C. Gross, J. Louis and G. Palma

SUSY BREAKING IN SUGRA

Constraints on realistic models

In a SUGRA model, the scalar potential V should allow for spontaneous SUSY breaking with certain non-trivial features.

- Phenomenology: To get a viable particle vacuum, need a point where $V \gtrsim 0$, $V' = 0$ and $V'' > 0$.
- Cosmology: To get a viable period of slow-roll inflation, need a region where $V > 0$, $V' \simeq 0$ and $V'' \gtrsim 0$.

The condition on V' can be satisfied by adjusting the values of the fields. But the conditions on V and V'' need an adjustment of parameters.

The natural question is then whether these two conditions can be used to restrict the class of models of potential interest. The answer is yes.

Algebraic formulation of the problem

Denef, Douglas 2005
Gomez-Reino, Scrucca 2006

Consider the critical situation where the scalar fields ϕ take values such that $V' = 0$ and leading to broken SUSY. The gravitino mass is $m_{3/2}$ and the Planck scale is set to 1.

The value of V is linked to SUSY breaking. This gives a first relevant parameter given by:

$$\gamma = \frac{V}{3\, m_{3/2}^2}$$

The value of V'' along a generic direction is not related to SUSY breaking and can be easily adjusted, whereas along the sGoldstino direction G it is related to SUSY breaking. This gives a second relevant parameter:

$$\lambda = \frac{V''(G)}{m_{3/2}^2}$$

The structure of SUGRA implies $\gamma \geq -1$ and most importantly that λ is constrained in terms of γ.

Necessary conditions

The requirements coming from phenomenology and cosmology imply that both at the final vacuum and in the rolling region one should have

$$\gamma \gtrsim 0$$

More quantitatively:

$$\gamma_{\rm vac} \ll 1\,, \quad \gamma_{\rm rol} \gg 1$$

Similarly, since λ defines bounds on the eigenvalues m^2 of V'', namely $\min(m^2) \leq \lambda\, m_{3/2}^2$ and $\max(m^2) \geq \lambda\, m_{3/2}^2$, one should also have, again both for vacuum metastablity and inflationary slow rolling:

$$\lambda \gtrsim 0$$

More quantitatively:

$$\lambda_{\rm vac} : \text{sizable}\,, \quad \lambda_{\rm rol} : \text{free}$$

GENERAL METASTABILITY CONSTRAINT

Models with chiral multiplets

A model with chiral multiplets Φ^i is specified by a real Kähler potential K and a holomorphic superpotential W. The Lagrangian depends only on

$$G = K + \log|W|^2$$

The value of G determines the gravitino mass scale:

$$m_{3/2} = e^{G/2}$$

The first derivatives of G determine the auxiliary fields:

$$F_i = -e^{G/2}\, G_i$$

The mixed second derivatives of G define the target-space geometry:

$$g_{i\bar{\jmath}} = G_{i\bar{\jmath}}$$

Critical points

The scalar potential takes the form

$$V = e^{G}\big(G^{k}G_{k} - 3\big)$$

Critical points are determined by the stationarity conditions

$$V_i = e^{G}\big(G_i + \nabla_i G_k G^k\big) + G_i V = 0$$

At such a point, the scalar mass matrix is given by

$$M^2_{IJ} = \begin{pmatrix} V_{i\bar{j}} & V_{ij} \\ V_{\bar{i}\bar{j}} & V_{\bar{i}\bar{j}} \end{pmatrix}$$

where

$$V_{i\bar{j}} = e^{G}\big(g_{i\bar{j}} + \nabla_i G_k \nabla_{\bar{j}} G^k - R_{i\bar{j}p\bar{q}} G^p \bar{G}^{\bar{q}}\big) + \big(g_{i\bar{j}} - G_i G_{\bar{j}}\big) V$$

$$V_{ij} = e^{G}\big(2\, G_{ij} + \nabla_i \nabla_j G_k G^k\big) + \big(G_{ij} - G_i G_j\big) V$$

Energy and sGoldstino mass

We assume now that $G^i \neq 0$ at the critical point, implying that SUSY is spontaneously broken.

The value of V in units of $m_{3/2}^2$ is given simply by:

$$\gamma = \frac{V}{3\,m_{3/2}^2} = -1 + \frac{1}{3}\,G^k G_k$$

The average of the values of $V''(G_\pm)$ in units of $m_{3/2}^2$ along the two real sGoldstino directions $G_+^I = (G^i, G^{\bar\imath})$ and $G_-^I = (iG^i, -iG^{\bar\imath})$ also takes a very simple form and is given by:

$$\lambda = \frac{V_{i\bar\jmath}\,G^i G^{\bar\jmath}}{m_{3/2}^2 G^k G_k} = 2 - \frac{R_{i\bar\jmath p\bar q}\,G^i \bar G^{\bar\jmath} G^p G^{\bar q}}{G^k G_k}$$

Imagine now that K is fixed whereas W is arbitrary. Then $g_{i\bar\jmath}$ and $R_{i\bar\jmath p\bar q}$ are fixed whereas G^i is arbitrary. We see that γ depends only on the norm of G^i, whereas λ also depends on the orientation of G^i.

Metastability constraint

Gomez-Reino, Scrucca 2006

For a given γ the value of λ depends on the sectional curvature along the normalized Goldstino direction $f^i = G^i/\sqrt{G^k G_k}$. One finds:

$$\lambda(f) = 3(1+\gamma)\Sigma(f) - 2\gamma$$

in terms of the shifted sectional curvature

$$\Sigma(f) = \frac{2}{3} - R_{i\bar{j}p\bar{q}}\, f^i f^{\bar{j}} f^p f^{\bar{q}}$$

We see that for $\gamma \gtrsim 0$ the necessary condition $\lambda \gtrsim 0$ implies that:

$$\Sigma(f) \gtrsim \frac{2}{3}\frac{\gamma}{1+\gamma} \gtrsim 0$$

We thus need to get $\Sigma(f) \gtrsim 0$ by suitably dialing f^i at the given point. In general $\Sigma(f) \leq \Sigma_{\max}$ and this implies a condition on the geometry:

$$\Sigma_{\max} \gtrsim \frac{2}{3}\frac{\gamma}{1+\gamma} \gtrsim 0$$

Bound on masses

Whenever $\boldsymbol{\Sigma_{\mathrm{max}}}$ is positive but finite, there is a bound on how large $\boldsymbol{\lambda}$ can be for a given $\boldsymbol{\gamma}$:

$$\boldsymbol{\lambda_{\mathrm{max}} = 3(1+\gamma)\Sigma_{\mathrm{max}} - 2\gamma}$$

This implies an upper bound on the mass of the lightest scalar:

$$\boldsymbol{\frac{m^2_{\mathrm{lightest}}}{m^2_{3/2}} \leq \lambda_{\mathrm{max}}}$$

Importance for cosmological history

Acharya, Kane, Kuflik 2010

The natural scale for the curvature in effective supergravity descriptions of string models is the Planck scale, corresponding to:

$$\boldsymbol{\Sigma_{\mathrm{max}} \sim 1}$$

This suggest a non-thermal cosmological history of the Universe.

CONSTANT CURVATURE SPACES

Maximally symmetric case

An other very simple case is the maximally symmetric case with

$$K = -r \log\left(1 - \sum_i \Phi^i \bar{\Phi}^i\right)$$

This corresponds to the following coset manifold

$$\mathcal{M} = \frac{SU(1,n)}{U(1) \times SU(n)}$$

The sectional curvature depends on r but not on the direction. One gets:

$$\Sigma = \frac{2}{3} - \frac{2}{r}$$

This trivially leads to

$$\Sigma_{\mathrm{max}} = \frac{2}{3}\frac{r-3}{r} \;\Rightarrow\; \text{positive when } r > 3$$

Minimally symmetric case

An other very simple example is the minimally symmetric case with:

$$K = -\sum_i r_i \log(1 - \Phi^i \bar{\Phi}^i) \quad \text{with} \quad \sum_i r_i = r$$

This corresponds to the following product of coset manifolds:

$$\mathcal{M} = \prod_i \frac{SU(1,1)}{U(1)}$$

The curvature depends on the r_i and $n-1$ angles parametrizing the S^n defined by $x_a = |e_i^a f^i|$ with $\sum_a x_a^2 = 1$. One finds:

$$\Sigma(x_a) = \frac{2}{3} - \sum_a \frac{2}{r_a} x_a^4$$

The maximum of this function occurs for $x_a = \sqrt{r_a/r}$ and is equal to

$$\Sigma_{\max} = \frac{2}{3}\frac{r-3}{r} \;\Rightarrow\; \text{positive when } r > 3$$

Less symmetric cases

On another interesting example is that of the less symmetric space with

$$K = -\frac{r}{2}\log\left(1 - 2\sum_i \Phi^i\bar{\Phi}^i + \sum_{ij}(\Phi^i\bar{\Phi}^j)^2\right)$$

This corresponds to the following coset manifold

$$\mathcal{M} = \frac{SO(2,2n-2)}{SO(2)\times SO(2n-2)}$$

The sectional curvature depends only on r and 1 angle parametrizing the S^1 defined by $x_\pm = \sqrt{1 \pm \sqrt{1 - |\sum_a (e^a{}_i f^i)^2|^2}}/\sqrt{2}$ with $x_+^2 + x_-^2 = 1$. One gets:

$$\Sigma(x_\pm) = \frac{2}{3} - \frac{4}{r}\left(x_+^4 + x_-^4\right)$$

The maximum of this function occurs for $x_\pm = 1/\sqrt{2}$ and is equal to

$$\Sigma_{\max} = \frac{2}{3}\frac{r-3}{r} \;\Rightarrow\; \text{positive when } r > 3$$

General result

Gomez-Reino, Scrucca 2007

For all the coset manifolds the value of $\Sigma_{\mathbf{max}}$ depends in a universal way on the parameter r defining the overall curvature scale through the relation

$$K^i K_i = r$$

One always finds:

$$\Sigma_{\mathbf{max}} = \frac{2}{3}\frac{r-3}{r} \;\Rightarrow\; \text{positive when } r > 3$$

MODULI IN STRING MODELS

General properties

At leading order in the week-coupling and low-energy expansions, the form of K for the moduli is fixed by the reduction of the kinetic terms. The corresponding sigma-model manifold $\mathcal{M}$ is related to the geometric moduli space of the space-time compactification manifold.

The general form of $\mathcal{M}$ always involves a factor spanned by the dilaton S and a factor spanned by one or several Kähler moduli T^i. Focusing on these fields, K involves a homogeneous function Y of degree 3:

$$K = -\log(S + \bar{S}) - \log Y(T^i + \bar{T}^i)$$

This corresponds to a manifold of the type

$$\mathcal{M} = \mathcal{M}_S \times \mathcal{M}_T$$

Dilaton domination

The dilaton cannot dominate, because its K is fixed and leads to

$$\Sigma_{\max} = -\frac{4}{3} < 0$$

Kähler moduli domination

The Kähler moduli may instead dominate, because their K is not fixed. However it satisfies the no-scale constraint $K^i K_i = 3$, which implies that the curvature along the direction $k^i = K^i/\sqrt{3}$ takes the value

$$R_{i\bar{\jmath}p\bar{q}} k^i k^{\bar{\jmath}} k^p k^{\bar{q}} = \frac{2}{3}$$

This implies that $\Sigma(k) = 0$. But nothing excludes the possibility that along other directions one may get $\Sigma(f) > 0$. We thus conclude that

$$\Sigma_{\max} \geq 0$$

Orbifold models

In orbifold models, $\mathcal{M}_T$ is always a simple coset manifold. The function $\Sigma(f)$ does not depend on the point and can be studied easily, as already seen. One finds that for $f^i \neq k^i$ the situation always gets worse:

$$\Sigma_{\max} = 0 \text{ in all cases}$$

Calabi-Yau models

In Calabi-Yau models, $\mathcal{M}_T$ is usually a non-coset manifold. The function $\Sigma(f)$ depends on the point and is more difficult to study. One finds that for $f^i \neq k^i$ the situation may either improve and worsen:

$$\Sigma_{\max} \begin{cases} = 0 \text{ in some cases} \\ > 0 \text{ in some cases} \end{cases}$$

Constraints on the curvature

Since K depends only on $T^i + \bar{T}^i$, we can use a real index notation. The degree 3 homogeneity of e^{-K} implies that $(T^i + \bar{T}^i)K_i = -3$, which is stronger than the no-scale condition $K^i K_i = 3$ and implies that:

$$\Gamma_{ijp} k^p = \frac{2}{\sqrt{3}}\, g_{ij} \qquad R_{ijpq} k^q = \frac{1}{\sqrt{3}} \Gamma_{ijp}$$

$$R_{ijpq} k^p k^q = \frac{2}{3}\, g_{ij} \qquad R_{ijpq} k^j k^q = \frac{2}{3}\, g_{ip}$$

Let us now decompose the Goldstino direction into two orthogonal pieces:

$$f^i = \cos\theta\, k^i + \sin\theta\, n^i$$

One then easily computes:

$$\Sigma(\theta, n) = \sin^4\theta \left[\frac{2}{3} + \frac{4}{3}\mathrm{ctg}^2\theta - \frac{4}{\sqrt{3}}\mathrm{ctg}\,\theta\, \Gamma_{ijp} n^i n^j n^p - R_{ijpq} n^i n^j n^p n^q\right]$$

Optimal Goldstino direction

Covi, Gomez-Reino, Gross, Louis, Palma, Scrucca 2008

To find the optimal Goldstino direction, it is convenient to rewrite Σ in the following form in terms of $P^{ij} = g^{ij} - k^i k^j$:

$$\Sigma(\theta, n) = \sin^4\theta \Big[\Big(\frac{2}{3} - R_{ijpq} n^i n^j n^p n^q + \frac{1}{2} \Gamma_{ijr} P^{rs} \Gamma_{pqr} n^i n^j n^p n^q \Big) - \frac{1}{2} \Big(\Gamma_{rij} n^i n^j - \frac{4}{\sqrt{3}} \mathrm{ctg}\, \theta\, n_r \Big) P^{rs} \Big(\Gamma_{spq} n^p n^q - \frac{4}{\sqrt{3}} \mathrm{ctg}\, \theta\, n_s \Big) \Big]$$

Given some n^i, Σ is maximized for some value of θ, but one must then find the optimal choice for the direction n^i to determine Σ_{max}. To get a positive result it is certainly necessary that the first term be positive.

To make further progress and compute Σ_{max}, one needs a more detailed knowledge of the form of K. Fortunately, this is well known for string models compactified on Calabi-Yau manifolds.

DETAILED ANALYSIS

Heteroric models

Dixon, Kaplunovsky, Louis 1990
Candelas, de la Ossa 1991

For a Calabi-Yau manifold with intersection numbers d_{ijk}, one finds:

$$K = -\log\left(\frac{4}{3} d_{ijk} t^i t^j t^k\right) \quad \text{with} \quad 2\, t^i = T^i + \bar{T}^i$$

It follows that:

$$g_{ij} = \sqrt{3}\, e^K d_{ijp} k^p + 3\, k_i k_j$$

$$\Gamma_{ijk} = -e^K d_{ijk} + \sqrt{3}\, (g_{ij} k_k + g_{ik} k_j + g_{jk} k_i) - 3\sqrt{3}\, k_i k_j k_k$$

$$R_{ijpq} = g_{ij} g_{pq} + g_{iq} g_{pj} - e^{2K} d_{ipr} g^{rs} d_{jqs}$$

We see that $\mathcal{M}_T$ is Special-Kähler and we deduce that:

$$P^{rs} \Gamma_{spq} n^p n^q = -\, e^K P^{rs} d_{spq} n^p n^q$$

$$R_{ijpq} n^i n^j n^p n^q = \frac{5}{3} - e^{2K} d_{ipr} P^{rs} d_{jqs} n^i n^j n^p n^q$$

Two-field models

Covi, Gomez-Reino, Gross, Louis, Palma, Scrucca 2008

In this case n^i is uniquely fixed and $P^{ij} = n^i n^j$. One then finds:

$$\Sigma(\theta) = \sin^4\theta \left[\left(\frac{3}{2}\alpha^2 - 1 \right) - \frac{1}{2} \left(\alpha - \frac{4}{\sqrt{3}} \mathrm{ctg}\,\theta \right)^2 \right]$$

where

$$\alpha = -e^K d_{ijk} n^i n^j n^k$$

We need the first term to be positive. But more explicitly one finds:

$$a = \frac{3}{2}\alpha^2 - 1 = -\frac{9}{8} e^{4K} \det{}^{-3} g_{ij}\, \Delta$$

in terms of the discriminant associated to the intersection numbers:

$$\Delta = -d_{111}^2 d_{222}^2 + 3\, d_{112}^2 d_{122}^2 - 4\, d_{111} d_{122}^3 - 4\, d_{112}^3 d_{222} + 6\, d_{111} d_{112} d_{122} d_{222}$$

Since the rest of a is negative definite, to get $a > 0$ we need:

$$\Delta < 0$$

Bounds on masses

Covi, Gomez-Reino, Gross, Palma, Scrucca 2009

Let us find the maximum of $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ over $\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{2}\boldsymbol{\pi}[$ defining the Goldstino direction, for any fixed $\boldsymbol{a} \in [\mathbf{0}, +\infty[$ depending on the point, with

$$\Sigma(\theta) = \sin^4\theta \left[a - \frac{8}{3}\left(\mathrm{ctg}\,\theta - \sqrt{\frac{1+a}{8}}\right)^2\right]$$

The maximum $\boldsymbol{\Sigma}_{\mathbf{max}}$ increases monotonically with $\boldsymbol{a}$, and one finds:

$$\Sigma_{\mathrm{max}} \simeq \begin{cases} \dfrac{64}{81}\,a \,, & a \ll 1 \\[2ex] \dfrac{2}{3}\,a \,, & a \gg 1 \end{cases} \quad \Rightarrow \quad \Sigma_{\mathrm{max}} < +\infty$$

We conclude that there is no bound on moduli masses:

$$\frac{m^2_{\mathrm{lightest}}}{m^2_{3/2}} < +\infty$$

Orientifold models

Grimm, Louis 2004

For a Calabi-Yau manifold with intersection numbers d^{ijk}, one finds:

$$K = -\log\left(\frac{1}{48} d^{ijk} t_i t_j t_k\right)^2 \quad \text{with} \quad \frac{1}{8} d^{ijk} t_j t_k = T^i + \bar{T}^i$$

It follows that:

$$\begin{aligned}
g_{ij} &= \sqrt{3}\, e^{-K} d_{ijp} k^p + 3\, k_i k_j \\
\Gamma_{ijk} &= e^{-K} d_{ijk} - \sqrt{3}\big(g_{ij} k_k + g_{ik} k_j + g_{jk} k_i\big) + 3\sqrt{3}\, k_i k_j k_k \\
R_{ijpq} &= -\, g_{iq} g_{pj} + e^{-2K}\big(d_{ijr} g^{rs} d_{pqs} + d_{ipr} g^{rs} d_{jqs}\big) \\
&\quad - \sqrt{3}\, e^{-K}\big(d_{ijp} k_q + \mathrm{p.}\big) + 3\big(g_{ij} k_p k_q + \mathrm{p.}\big) + 9\, k_i k_j k_p k_q
\end{aligned}$$

We see that $\mathcal{M}_T$ is Kähler and we deduce that:

$$\begin{aligned}
P^{rs} \Gamma_{spq} n^p n^q &= e^{-K} P^{rs} d_{spq} n^p n^q \\
R_{ijpq} n^i n^j n^p n^q &= -\frac{1}{3} + 2\, e^{-2K} d_{ipr} P^{rs} d_{jqs} n^i n^j n^p n^q
\end{aligned}$$

Two-field models

Covi, Gomez-Reino, Gross, Louis, Palma, Scrucca 2008

In this case n^i is uniquely fixed and $P^{ij} = n^i n^j$. One then finds:

$$\Sigma(\theta) = \sin^4\theta \left[\left(1 - \frac{3}{2}\alpha^2\right) - \frac{1}{2}\left(\alpha - \frac{4}{\sqrt{3}}\mathrm{ctg}\,\theta\right)^2\right]$$

where α depends only on the point and is given by

$$\alpha = e^{-K} d_{ijk} n^i n^j n^k$$

We need the first term to be positive. But more explicitly one finds:

$$a = 1 - \frac{3}{2}\alpha^2 = \frac{9}{8}e^{-4K}\det{}^3 g_{ij}\,\Delta$$

in terms of the discriminant associated to the intersection numbers:

$$\Delta = -d^{111\,2}d^{222\,2} + 3\,d^{112\,2}d^{122\,2} - 4\,d^{111}d^{122\,3} - 4\,d^{112\,3}d^{222} + 6\,d^{111}d^{112}d^{122}d^{222}$$

Since the rest of a is positive definite, to get $a > 0$ we need

$$\Delta > 0$$

Bounds on masses

Covi, Gomez-Reino, Gross, Palma, Scrucca 2009

Let us find the extremum of $\Sigma(\theta)$ over $\theta \in [0, 2\pi[$ defining the Goldstino direction, for any fixed $a \in [0, 1]$ depending on the point, with

$$\Sigma(\theta) = \sin^4\theta \left[a - \frac{8}{3} \left(\mathrm{ctg}\,\theta - \sqrt{\frac{1-a}{8}} \right)^2 \right]$$

The maximum Σ_{max} increases monotonically with a, and one finds:

$$\Sigma_{\mathrm{max}} \simeq \begin{cases} \frac{64}{81}\, a \;, & a \ll 1 \\ 1 \;, & a \to 1 \end{cases} \quad \Rightarrow \quad \Sigma_{\mathrm{max}} \leq 1$$

We conclude that there is a bound on moduli masses:

$$\frac{m^2_{\mathrm{lightest}}}{m^2_{3/2}} \leq 3 + \gamma$$

CONCLUSIONS

- In SUGRA theories, there is a strong necessary condition on K for the existence of metastable de Sitter vacua or slow-roll inflationary regions, independently of the form of W.
- In string theories, one can apply this result to the moduli sector and derive topological constraints on the Calabi-Yau manifolds and bounds on the possible values of moduli masses.