Playing with diagrams

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Plan of the lectures

Introduction

Amplitudes and S-matrix elements (amplitudes with the external particles on mass shell) are defined in terms of diagrams.

Properties of the theory such as unitarity, causality, symmetry are a consequence of the properties of the diagrams.

Since the diagram method is a perturbative method all of these are thus defined insofar perturbation theory is valid.

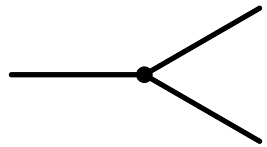
Certain non-perturbative results (unstable particles, bound states) can nonetheless be studied using partial summation.

Furthermore, the renormalization group can be established and scaling equations derived. These may also have meaning beyond perturbation theory,

Definitions

Simplest theory: one kind of particle, one 3-point vertex.

Propagator: 

Vertex: 

$$\Delta_F(x_2 - x_1) = \frac{1}{(2\pi)^4 i} \int d_4p \; e^{ip(x_2-x_1)} \frac{1}{p^2 + m^2 - i\epsilon}$$

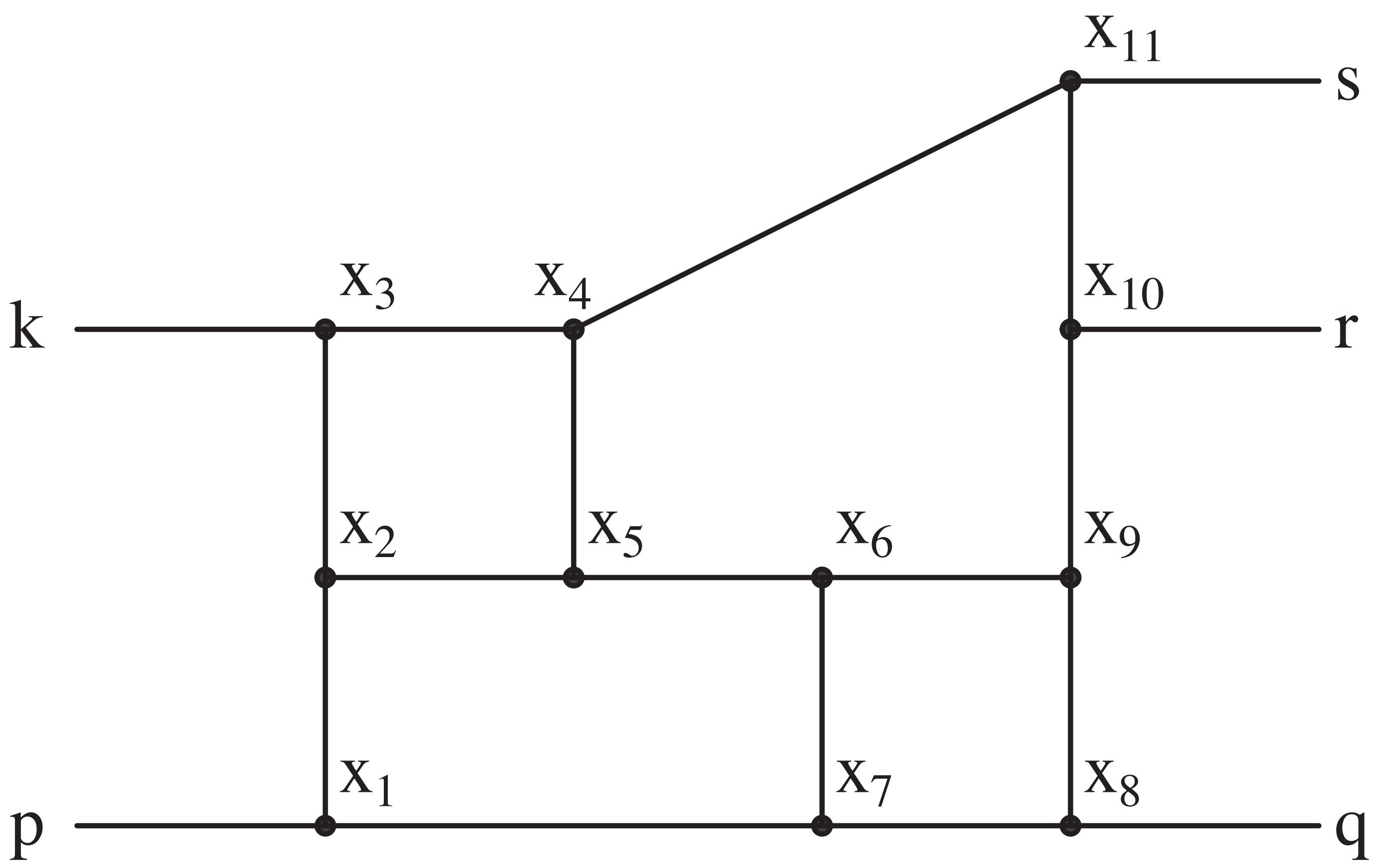
Vertex factor: ig

External lines: $\dfrac{1}{\sqrt{2p_0}} \, e^{ipx}$. 

$p =$ ingoing momentum.

Diagram: points $x_1 ... x_n$, vertex in each point, connected by propagators or with external line attached. All x to be integrated over all space-time.

Example of a diagram



$$\frac{(ig)^{11} e^{ipx_1} e^{ikx_2} e^{-iqx_8} \ldots}{\sqrt{32 p_0 k_0 q_0 r_0 s_0}} \; \Delta_F(x_2 - x_1) \; \Delta_F(x_3 - x_2) \ldots$$

All x to be integrated over all space-time.

The θ-functions

$$\Delta^{\pm}(x) = \frac{1}{(2\pi)^3} \int d_4p \; e^{ipx} \theta(\pm p_0) \delta(p^2 + m^2)$$

$$\Delta_F(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x)$$

$$\Delta_F^*(x) = \theta(x_0)\Delta^-(x) + \theta(-x_0)\Delta^+(x)$$

$\Delta^+(x_2 - x_1)$ 

$\Delta^-(x_2 - x_1)$ 

$\Delta_F^*(x_2 - x_1)$ 

Minus sign for every circle.

Simple consequence:

$+$

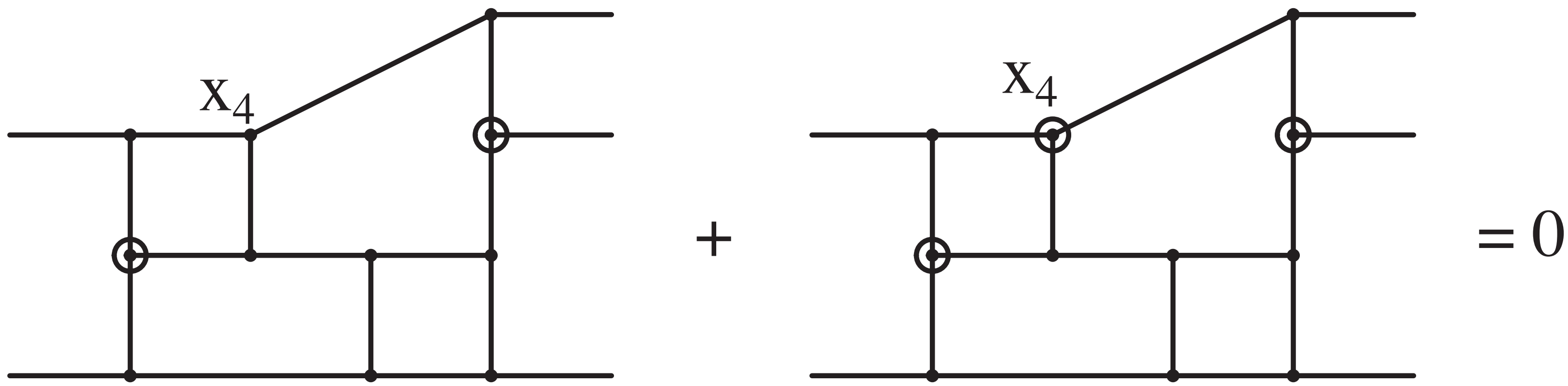

$-$


$-$

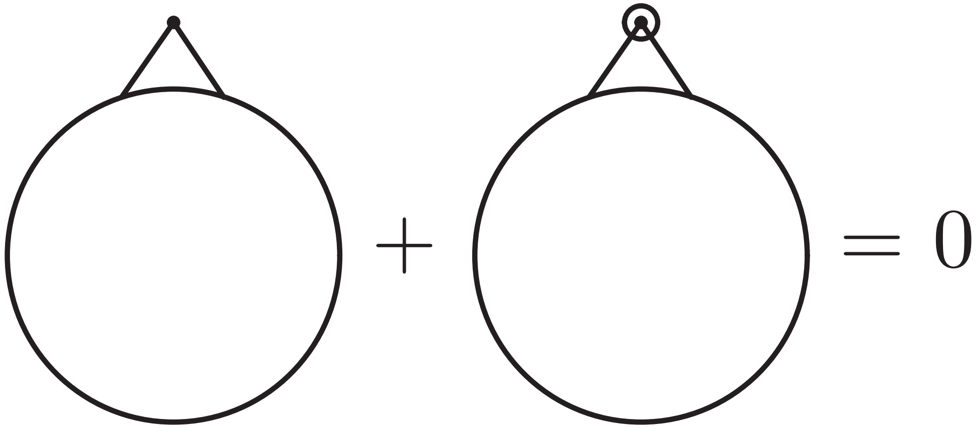

$= 0$

$$\Delta_F(x) + \Delta_F^*(x) - \Delta^+(x) - \Delta^-(x) = 0$$

The largest time equation

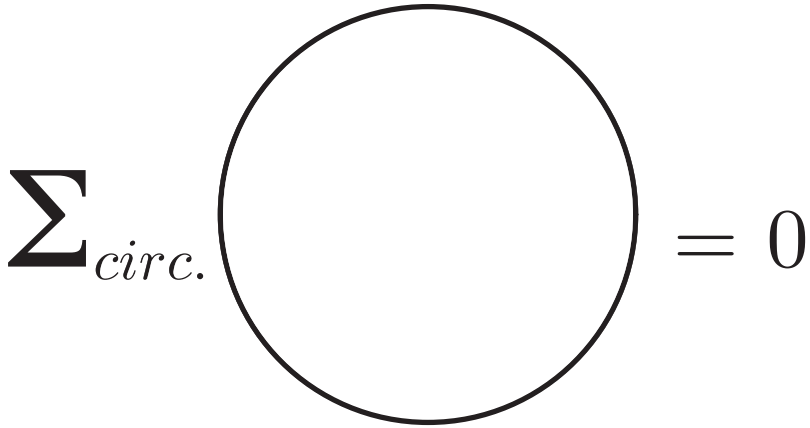


The above diagrammatic equation holds if $(x_4)_0$ is larger than any other time component $(x_n)_0, n \neq 4$.
In general:

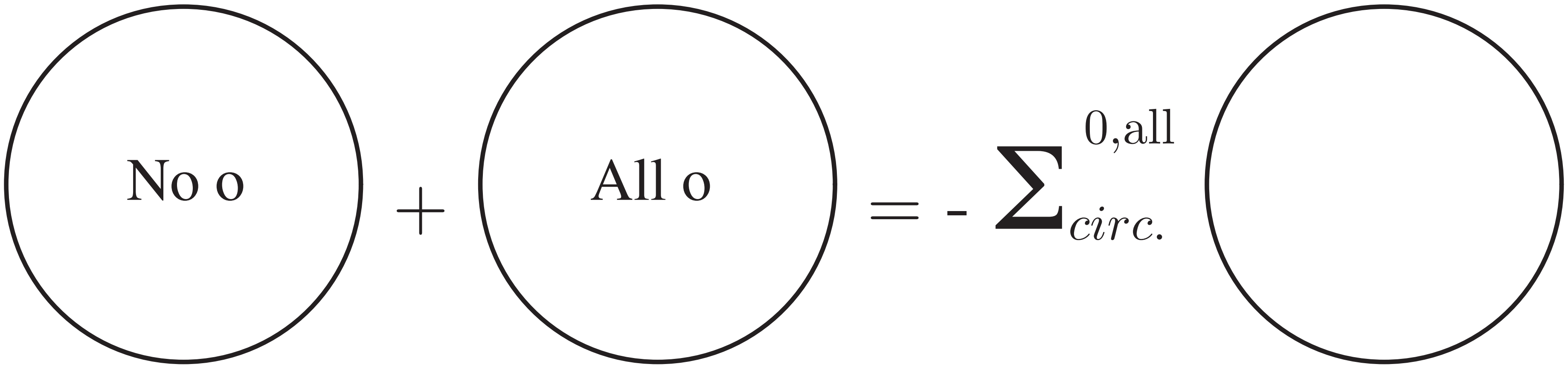


for any diagram with any number of circled vertices.

The unitarity equation



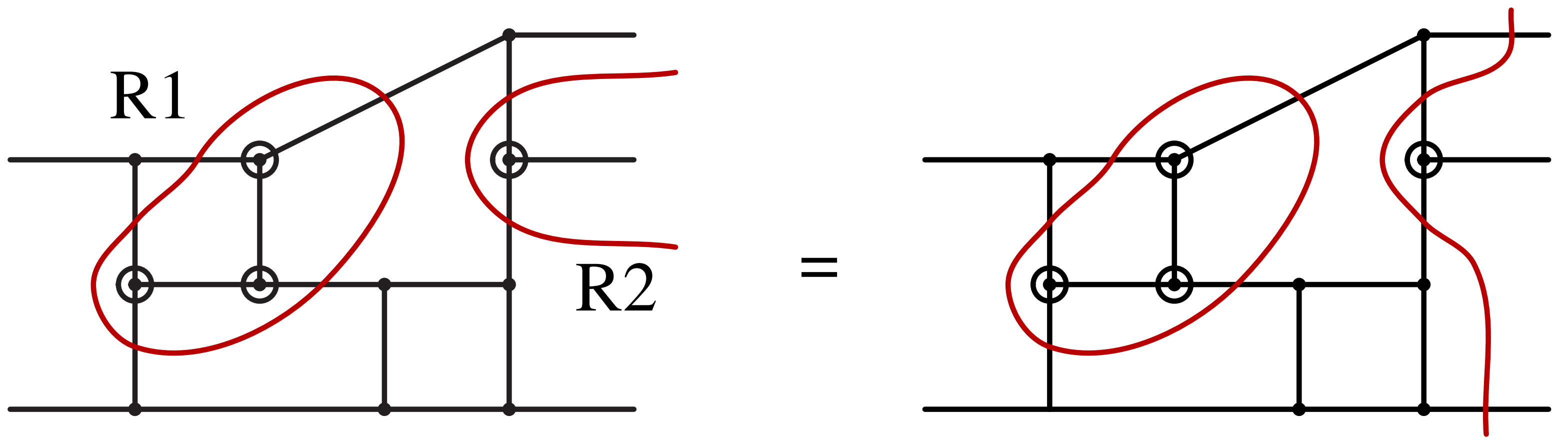
This may be split:



Left side is twice ‘imaginary’ part of the diagram.
This is the unitarity equation.

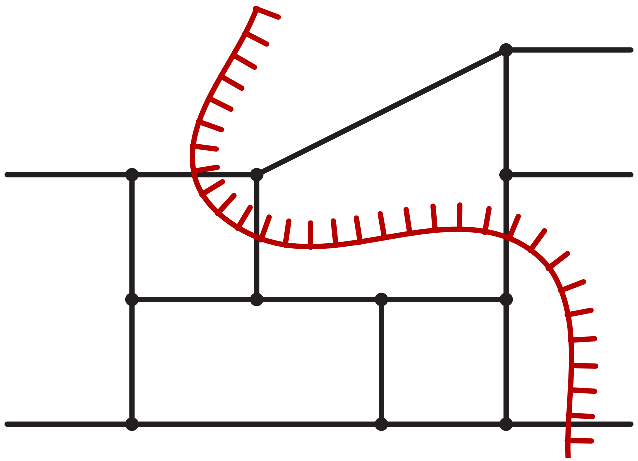
Regions

A region is defined as a group of circled (or uncircled) connected vertices. Example:

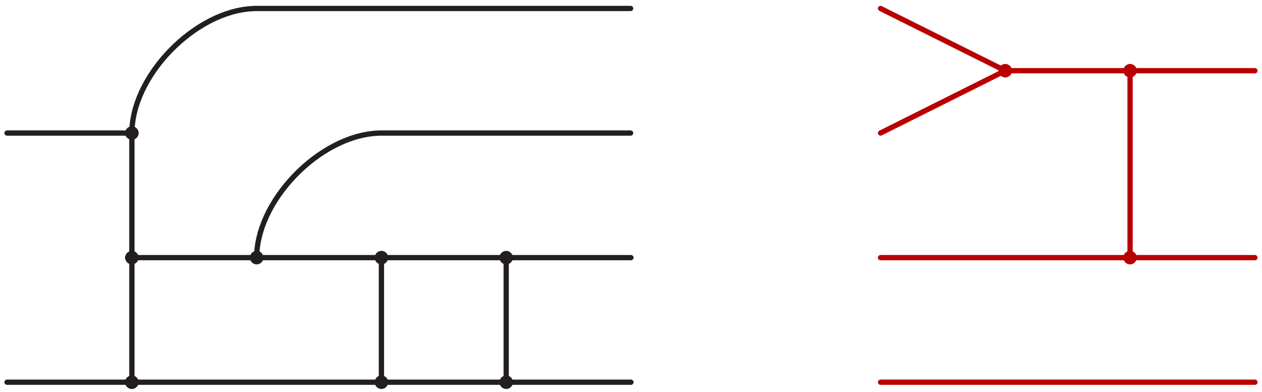


A line crossed by a red line is a Δ^+ (or Δ^-). Going over to the momentum representation by doing the x-integrals all of these lines have a $\theta(p_0)$ such that p_0 flows inward only. A diagram containing enclosed regions such as R1 are zero. As R2 has a line carrying momentum outwards such a region does not imply a zero diagram.

Cut diagram



It can be seen as two diagrams glued together:



This is the product of some diagram with some diagram*, with connecting lines integrated over phase space: $\int d_4p\ \theta(p_0)\delta(p^2 + M^2)$.

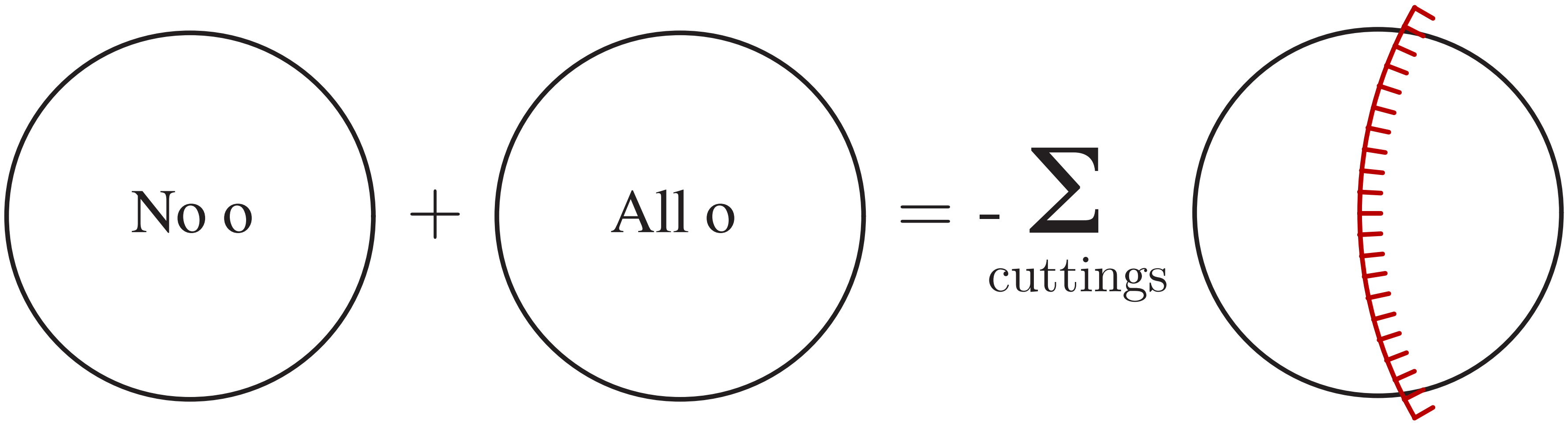
Cutting Rules

The S-matrix S must be Unitary. This means $S\ S^\dagger = 1$. The T-matrix is defined by $S = 1 + iT$ and $S^\dagger = 1 - iT^\dagger$. Thus

$$S\ S^\dagger = (1 + iT)(1 - iT^\dagger) = 1 + i(T - T^\dagger) + T\ T^\dagger = 1$$

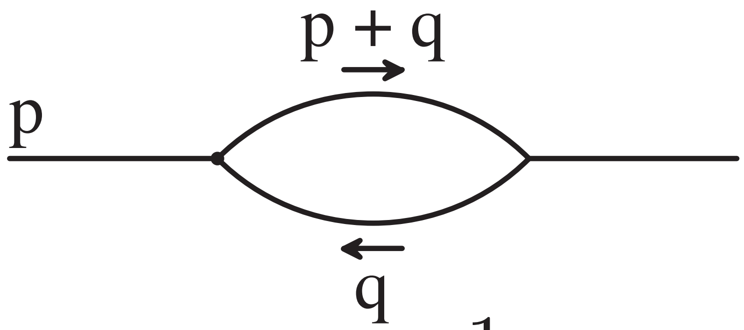
$$2i\ Im(T) = -T\ T^\dagger$$

The left and right hand side correspond to the diagrammatic equation:



The blobs now stand for the sum of all diagrams with a given number of external lines.

Example: self-energy diagram



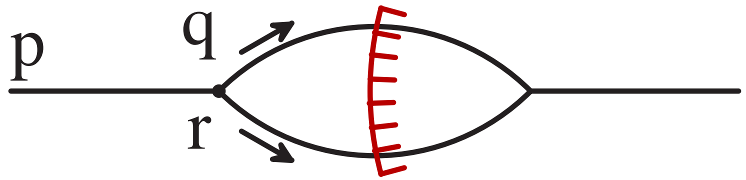
$$g^2 \int d_4q \; \frac{1}{(q^2 + M^2 - i\epsilon)((p+q)^2 + M^2 - i\epsilon)} =$$

$$i\pi^2 g^2 \left[C - \int_0^1 dx \;\; \ln(x(1-x)p^2 + M^2 - i\epsilon) \right]$$

To get the "imaginary" part see where the arg. of the ln is neg. Multiply length of the region by $-i\pi$.

$$2(\text{Im. part}) = -2\pi^3 g^2 \theta(-p^2 - 4M^2)\sqrt{1 + \frac{4M^2}{p^2}}$$

The right hand side of the cutting equation:



p restframe

$$-(2\pi g)^2 \int d_4q \int d_4r \delta_4()\theta(q_0)\delta(q^2+M^2)\theta(r_0)\delta(r^2+M^2)$$

$$= -(2\pi g)^2 \int \frac{d_3q}{2q_0}\frac{d_3r}{2r_0}\,\delta_4(p-q-r)$$

$$q_0 = \sqrt{\vec{q}^{\,2} + M^2}$$

$$r_0 = \sqrt{\vec{r}^{\,2} + M^2}$$

$$= -(2\pi g)^2 \int d_3q \, \frac{\delta(p_0 - q_0 - r_0)}{4q_0\sqrt{\vec{q}^{\,2} + M^2}}$$

$$= -(2\pi g)^2 \, \frac{\pi}{2} \, \frac{\theta(-p^2 - 4M^2)}{-p^2}\sqrt{(p^2+2M^2)^2 - 4M^4}$$

Unitarity

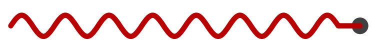
When is an S-matrix unitary ?

- All possible diagrams must be there
- The vertices must be real (apart from the explicitly mentioned i)
- For particles with spin the propagator must contain the spinsums.
- If the propagator does not completely agree with the spinsums there must be Ward identities correcting the differences
- If there are ghost particles not appearing as in- or outgoing lines there must be Ward identities

In theories with spin 1 or spin 2 particles Ward identities play an important role. In particular this is the case for massless particles since there the propagator does not correspond to the sum over the spins of the polarization vectors.

Ward identities usually express the symmetry content of the theory. Also if there is a partial symmetry breaking Ward identities can be derived.

The Photon

External photon lines: $\dfrac{e_\mu(k)}{\sqrt{2k_0}}\, e^{ikx}$. 

Polarization vector orthogonal to k: $e_\mu k_\mu = 0$

If k is taken along the z-axis, $k = (0, 0, \mathbf{k}, i\mathbf{k})$, thus $k^2 = 0$, there are three solutions:

$$e^1 = (1, 0, 0, 0), \quad e^2 = (0, 1, 0, 0) \quad e^3 = (0, 0, 1, i).$$

The longitudinal polarization vector e_3 cannot be normalized to 1, as $e^3 e^3 = 0$. It is an unphysical state and must be excluded. The trouble is that this is not a Lorentz invariant statement. To see this perform a Lorentz tranformation along the 1-axis followed by a rotation around the 2 axis and consider the tranformation of the four-vector k.

$$\begin{pmatrix} \beta & 0 & v & 0 \\ 0 & 1 & 0 & 0 \\ -v & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\beta & 0 & 0 & iv/\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -iv/\beta & 0 & 0 & 1/\beta \end{pmatrix}$$

$$\mathbf{k}(0,0,1,i) \rightarrow \mathbf{k}(-v/\beta,0,1,1/\beta) \rightarrow \mathbf{k}(0,0,1/\beta,i/\beta)$$

Thus k is transformed into a multiple of itself. This transformation applied to e_1 gives:

$$(1,0,0,0) \rightarrow (1,0,0,-iv)/\beta \rightarrow (1,0,-v/\beta,-iv/\beta)$$

Thus $e^1 \rightarrow e^1 - v/\beta \; e^3$. Note that $e^3 \rightarrow e^3/\beta$. Only one solution: the S-matrix must be such that the unphysical states do not scatter, i.e. a photon with polarization vector e^3 gives a zero S-matrix element. A symmetry and Ward identities are needed. Let us put that in diagrams.

The needed expression is

$$\sum_{j=1,2} e^j_\mu e^j_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \delta_{\mu\nu} - \frac{\overline{k}_\mu k_\nu + k_\mu \overline{k}_\nu}{(k\overline{k})}$$

where $k = (0, 0, \mathbf{k}, i\mathbf{k})$ and $\overline{k} = (0, 0, -\mathbf{k}, i\mathbf{k})$. This equation is rotation invariant, but not Lorentz invariant. Unitarity requires that the photon propagator has this in the numerator. Problems with Lorentz invariance. Need Ward identities.

Lagrangian and Feynman rules

Feynman rules are defined by specifying a Lagrangian. The recipe is:

• The propagator corresponds to the part of the Lagrangian quadratic in the fields. The propagator is simply minus the inverse of the fouriertransform of the quadratic part. A term $-i\epsilon$ must be added to define behaviour for zero denominator.

• All other terms (including terms linear in the fields) correspond to a vertex whose value can be read off directly from the Lagrangian. Conservation of energy and momentum is guaranteed by a δ function.

The matter of fourier transform means esentially that a factor ik_μ corresponds to ∂_μ where k is the momentum of the field on which ∂ operates.

For n identical fields a factor $n!$ must be added.

Example:

$$\mathcal{L} = -\tfrac{1}{2}\varphi\partial^2\varphi - \tfrac{1}{2}m^2\varphi^2 + g\varphi^3$$

The φ propagator and the vertex are:

$$\frac{1}{k^2 + m^2 - i\epsilon} \quad \text{and} \quad 6g\delta()$$

Note that partial integration in the Lagrangian is permitted. For a propagator one has that $(\partial\varphi)(\partial\varphi)$ gives the same as $-\varphi\partial^2\varphi$. The momentum of the first field is the opposite of the second, since also for the propagator one has conservation of momentum.

With these prescriptions the theory will have the usual properties with respect to transformations of the Lagrangian. For example, making the (canonical) transformation $\varphi \to \lambda\varphi$ gives no change in the S-matrix. This particular case is obvious: the propagator gets a factor $1/\lambda^2$, but every field in a vertex gets a factor λ. The factor λ cancels out. There is some question about what to do for external lines; that will not be discussed here. It requires a discussion about normalization of initial and final states.

Ward identities.

Lagrangian for QED in Lorentz gauge.

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \mathcal{L}_{elec.} - \tfrac{1}{2}(\partial_\mu A_\mu)^2$$
$$= -\tfrac{1}{2}\partial_\nu A_\mu \partial_\nu A_\mu + \tfrac{1}{2}(\partial_\mu A_\mu)^2 + \mathcal{L}_{elec.} - \tfrac{1}{2}(\partial_\mu A_\mu)^2$$

The second and last term cancel; to this Lagrangian corresponds the simple photon propagator

$$\frac{\delta_{\mu\nu}}{k^2 - i\epsilon.}$$

Except for the last term the Lagrangian is invariant under $A_\mu \to A_\mu - \lambda\partial_\mu\varphi$ (and the corresponding tranformation of the electron field in $\mathcal{L}_{elec}$).

Here now the essential trick. We take for φ a field, and more specifically a non-interacting field.

The S-matrix remains unchanged after a canonical transformation to all orders in λ. Note that what we introduced as a gauge transformation is now considered a canonical transformation that leaves the theory unchanged. In this context one must transform everything including the above mentioned non-gauge invariant term. Then in particular φ remains a free field.

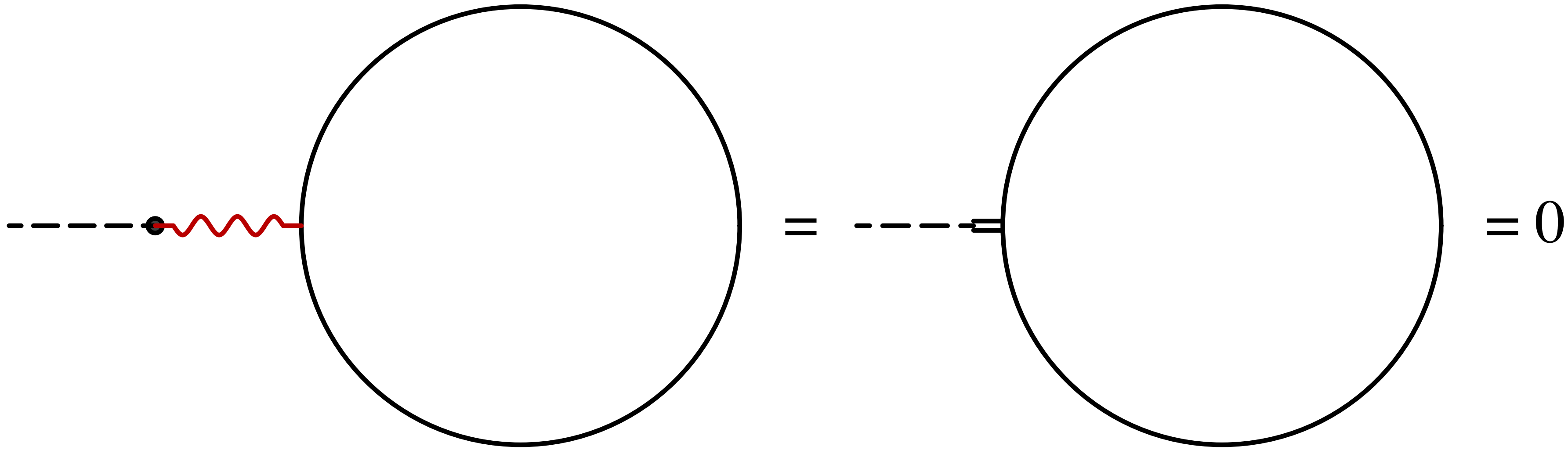
Under a gauge transformation $A_\mu \to A_\mu - \lambda \partial_\mu \varphi$:

$$\mathcal{L} \to \mathcal{L} + \lambda \partial_\mu A_\mu \partial^2 \varphi$$

One new vertex: $ik^2 \lambda k_\mu$

Any S-matrix element with one ingoing φ particle must be zero. That particle always couples through this vertex and one photon propagator.

An ingoing φ particle plus one photon propagator is $i\lambda k_\mu$. It follows:



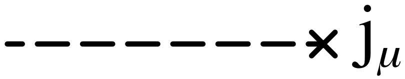
The short double line means connection to some vertex instead of a photon with a factor $i\lambda k_\mu$. Now the blob stands for the sum of all diagrams with some given number of external lines all on masshell.

Apart from the irrelevant factor $i\lambda$ this is the desired Ward identity for the case of one ingoing unphysical photon for which e^3_μ is proportional to k_μ.

Identities needed for multiple φ lines; no problem. But also for outgoing photon lines not on mass-shell (other propagators). To study that case add to the Lagrangian a term coupling the em field to an unspecified external source j_μ.

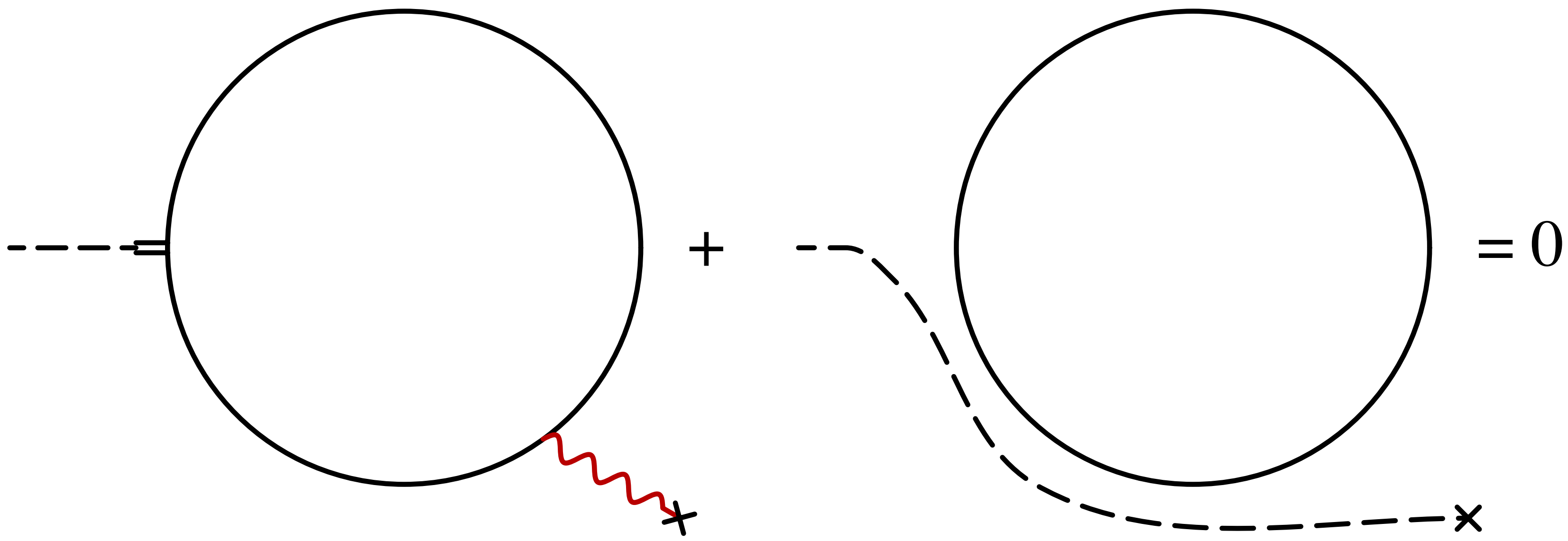
$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \mathcal{L}_{elec.} - \frac{1}{2}(\partial_\mu A_\mu)^2 + A_\mu j_\mu$$

After the gauge transformation we have the extra term $-\lambda\partial_\mu\varphi j_\mu$ and the vertex

$-i\lambda p_\mu$ 

and of course 

Free φ equation for one φ and one j:



The first blob includes the case that the photon line that φ was coupled to is directly connected to the source j. That cancels against the second blob. The original Ward identity still holds.

This makes clear that the extra terms in the photon propagator required by unitarity can be added without changing the S-matrix.

In summary, the free field technique is a very powerful technique that can notably also be used in case of a partially symmetric Lagrangian. Identities, called generalized Ward identities, may be derived that exploit the partial symmetry. The judicious use of currents (introduced in a systematic way by Schwinger) may be very helpful. Cases were this has been applied is the massive Yang-Mills theory (mass not due to a Higgs mechanism), and the well-known U1 problem (η mass).

Massive vector particles

A massive vector particle at rest has three polarization states. The corresponding polarization vectors are in the restframe, $k = (0, 0, 0, iM)$:

$$e^1 = (1, 0, 0, 0), e^2 = (0, 1, 0, 0), e^3 = (0, 0, 1, 0)$$

These are linear polarization vectors. Alternatively:

$$e^1 = \frac{1}{\sqrt{2}}(1, i, 0, 0), e^2 = \frac{1}{\sqrt{2}}(1, -i, 0, 0), e^3 = (0, 0, 1, 0)$$

These correspond to spin along the z-axis. They transform into themselves when rotating around that axis. All e satisfy the equation $k_\mu e^i_\mu(k) = 0$.

We need the expression encountered in the Unitarity equation, namely $\Sigma_i e^i_\mu (e^i_\nu)^\dagger$.

$$\Sigma_i \; e^i_\mu (e^i_\nu)^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{matrix} \delta_{\mu\nu} \text{ if } \mu, \nu \neq 4 \\ 0 \text{ if } \mu, \nu = 4. \end{matrix}$$

This may be written as $\delta_{\mu\nu} + k_\mu k_\nu / M^2$. In the case of massive photons (massive QED) the Ward identities remain true, and the $k_\mu k_\nu$ terms may be dropped. In the non-abelian case the Ward identities are much more complicated, and the $k_\mu k_\nu$ term does not give 0, but there is a remnant (behaving for large k_μ as a constant if a gauge symmetry holds).

Spin two particles. Massless and massive graviton

The polarization vectors are symmetric traceless 2-tensors. In addition $k_\mu e_{\mu\nu} = e_{\mu\nu} k_\nu = 0$.

$$e^1 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^2 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^3 = \sqrt{\tfrac{2}{3}} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad e^4 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$e^5 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Last row and column are 0

The corresponding expression is:

$$
\begin{aligned}
e^i_{\mu\nu} e^i_{\alpha\beta} = & \tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}) \\
& + \tfrac{1}{2}\left(\delta_{\mu\alpha}\frac{k_\nu k_\nu}{M^2} + \delta_{\nu\beta}\frac{k_\mu k_\alpha}{M^2} + \delta_{\mu\beta}\frac{k_\nu k_\alpha}{M^2} + \delta_{\nu\alpha}\frac{k_\mu k_\beta}{M^2}\right) \\
& + \tfrac{2}{3}\left(\tfrac{1}{2}\delta_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)\left(\tfrac{1}{2}\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{M^2}\right)
\end{aligned}
$$

If k is on mass shell then, going to the k restframe, one has that this is non-zero only if $\mu...\beta = 1, 2, 3$ This expression must be used in the propagator. Of course then k is no longer on mass shell. Assuming that the k_μ, k_ν etc. terms are zero because of symmetry we have for the propagator numerator:

$$
P^m_{\mu\nu\alpha\beta} = \tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) - \tfrac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta} \quad \mu...\beta = 1...4.
$$

For massless spin 2 particles only e^1 and e^2 apply. One finds now in the frame where k is along the third axis:

$\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})$ if $\mu ... \beta = 1, 2$, else zero.

Again, like in the photon case, this may be rewritten as an expression valid for all values of the indices by using instead of $\delta_{\mu\nu}$ (and similarly all other δ's) the expression:

$$\delta_{\mu\nu} - \frac{\bar{k}_\mu k_\nu + k_\mu \bar{k}_\nu}{(k\bar{k})}$$

This then must be used in the graviton propagator, also for k not on the mass shell. Assuming that symmetry takes care of all terms proportional to k_μ we are left with the numerator for propagator for the massless case:

$$P^0_{\mu\nu\alpha\beta} = \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta}),\ \mu...\beta = 1...4$$

Note the difference with the massive case:

$$P^m_{\mu\nu\alpha\beta} = \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) - \frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta},\ \ \mu...\beta = 1...4.$$

To compare with experiment one must consider two processes:

- Scattering of two massive objects. That must be according to Newtons law. That fixes the value of the coupling constant (graviton to particle).

- Scattering of light by a massive object.

This must be done for both the masssive and massless case. Insisting that Newton's law is the same for both cases one finds that the photon scattering cases are different by a factor 3/4.

Thus the scattering of light by the sun is different for the massless and massive case. Experiment tells us that the massless case applies. A massive graviton, no matter how small the mass, is excluded.

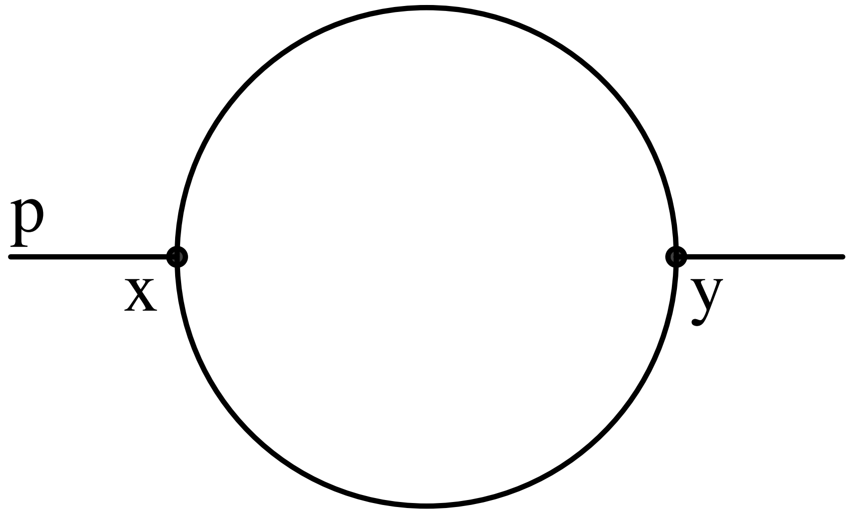
See van Dam and M.V., references.

Causality and dispersion relations.

In field theory causality enters in a somewhat abstract way. The basic idea is already contained in the definition of the propagator:

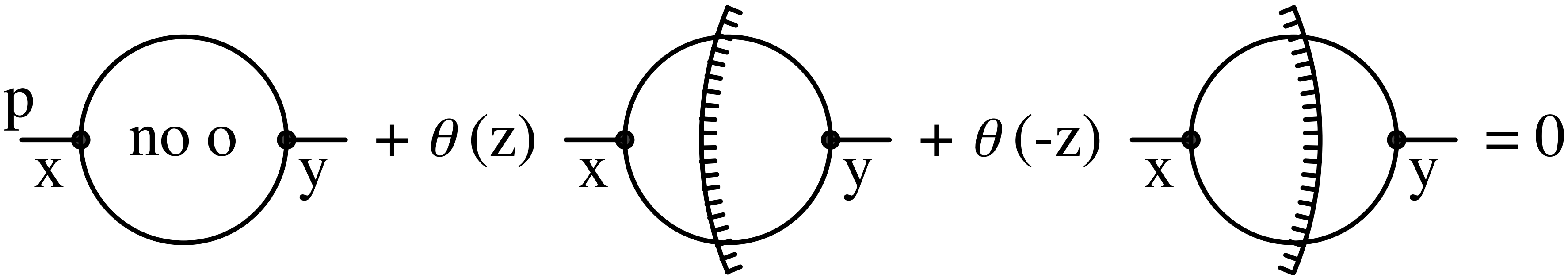
$$\Delta_F(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x)$$

In words: the energy flows in the direction of the largest time. Can be generalized. Two point case.

$\sum_{circ.}^{x}$ 

$= 0$ if $y_0 > x_0$.

Similar equation if $x_0 > y_0$. Adding the two equations and separating the parts without circles gives:



where z=x-y and as before region considerations were applied. Take the fouriertransform of this equation using:

$$\theta(z) = \frac{1}{2i\pi} \int dk_0 \frac{e^{ik_0 z_0}}{k_0 - i\epsilon} = \frac{1}{2\pi i} \int d_4 k \frac{e^{ik_0 z_0}}{k_0 - i\epsilon} \delta(\overrightarrow{k})$$

Denote the fourier transform of the leftmost term in the graphic equation above by $G(p)$.

$$G(p) = \frac{1}{\pi} \int \frac{dk_0}{k_0 - i\epsilon} \left[\theta(a_0) h(a) + \theta(b_0) h(b)\right]$$

with $a_0 = k_0 + p_0 - \sqrt{\overrightarrow{p}^2}$, $a = (p_0 + k_0)^2 - \overrightarrow{p}^2$, and $b_0 = k_0 - p_0 - \sqrt{\overrightarrow{p}^2}$, $b = (p_0 - k_0)^2 - \overrightarrow{p}^2$.

The function h represents the value of the cut graphs. It is in fact the imaginary part of the graphs. The θ functions explicitly show that the cut graphs are nonzero only if the energy is sufficiently large in order to have a state with particles on mass shell. For example, if there was a single particle of mass M in the intermediary state then one would have that its energy, $p_0 + k_0$, is $\sqrt{\overrightarrow{p}^2 + M^2}$. Forcing that $p_0 + k_0$ is larger than $\sqrt{\overrightarrow{p}^2}$ is a conservative way of putting this. h is a (Lorents invariant) function of the four-momentum squared, i.e. a and b. Note that one must sum over all intermediate states including integration over all phase space, so there is no further momentum dependence except on the overall momentum a (and b).

New variable $t = (p_0 + k_0)^2 - \overrightarrow{p}^2$. Expressing k_0 in t one has two roots; the θ functions forces the relation $k_0 = -p_0 + \sqrt{\overrightarrow{p}^2 + t}$ in the first term and $k_0 = -p_0 - \sqrt{\overrightarrow{p}^2 + t}$ in the second. Result:

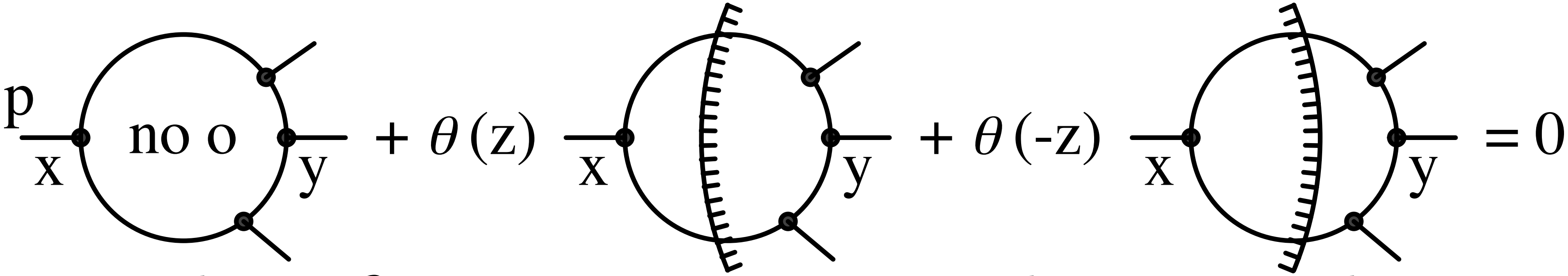
$$G(p) = \frac{1}{\pi} \int_0^\infty \frac{dt}{2\sqrt{\overrightarrow{p}^2 + t}} \left[\frac{1}{-p_0 + \sqrt{\overrightarrow{p}^2 + t - i\epsilon}} + \frac{1}{p_0 + \sqrt{\overrightarrow{p}^2 + t - i\epsilon}} \right] h(t)$$

$$= \frac{1}{\pi} \int \frac{dt}{t + p^2 - i\epsilon} h(t)$$

This is a dispersion relation.

Sidewise dispersion relations

In case of extra outgoing lines a similar treatment may be carried through.

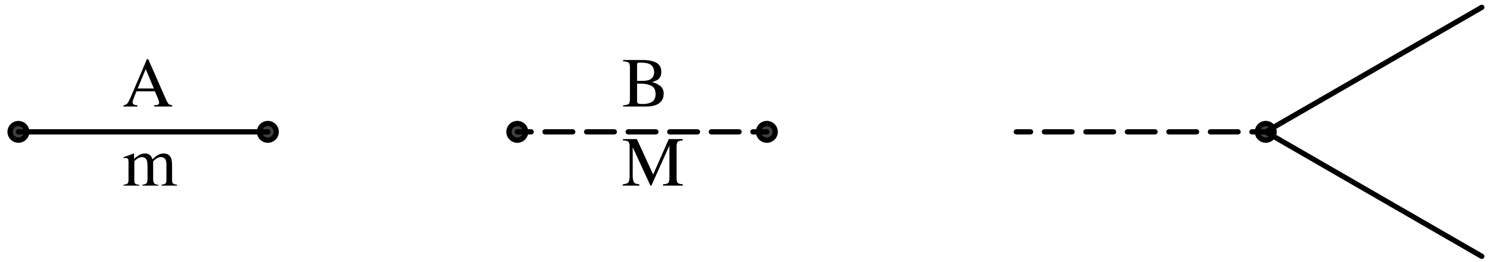


Using the infinite momentum technique a dispersion relation in p^2 can be derived. This is called a sidewise dispersion relation. It is used in calculation of complicated Feynman graphs, in for example the radiative corrections to the muon anomalous magnetic moment.

Remiddi, in very clear paper, has reviewed the largest time equation etc. In addition sidewise dispersion relations are introduced and discussed. See references.

Unstable Particles

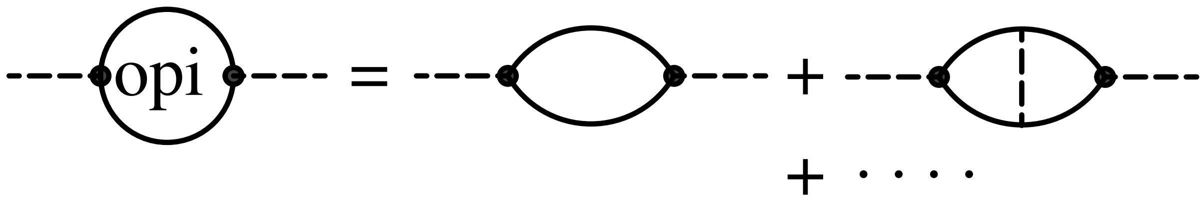
Two scalar particles A and B with masses m and M respectively, and $M > 2m$. Vertex with one A and two B. Feynman rules:



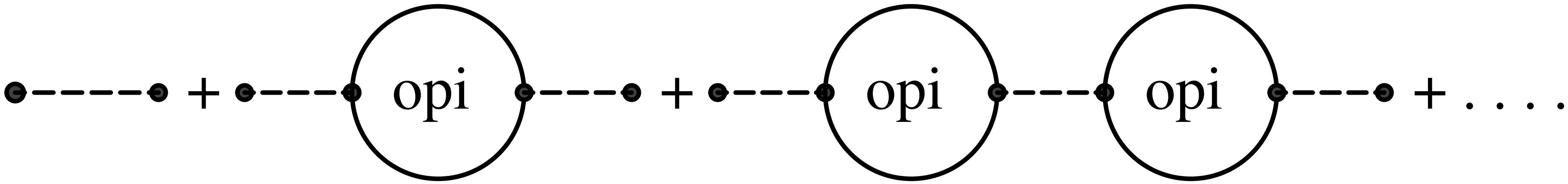
$$\frac{1}{p^2 + m^2 - i\epsilon} \qquad \frac{1}{q^2 + M^2 - i\epsilon} \qquad ig$$

B will be unstable. Not occurring as in- or out-going line of the S matrix. However, with these Feynman rules there will be cut B-lines. Unitarity ?

Solution: summation of B opi self-energy diagrams.



$F(p^2)$ = sum of one-particle irreducible diagrams.
Now make the Dyson summation:



The result is the "dressed" propagator $\overline{\Delta}_F^B$:

$$\overline{\Delta}_F^B = \Delta_F^B + \Delta_F^B F(p^2) \Delta_F^B + \Delta_F^B F(p^2) \Delta_F^B F(p^2) \Delta_F^B + ...$$

This is a geometric series. Sum:

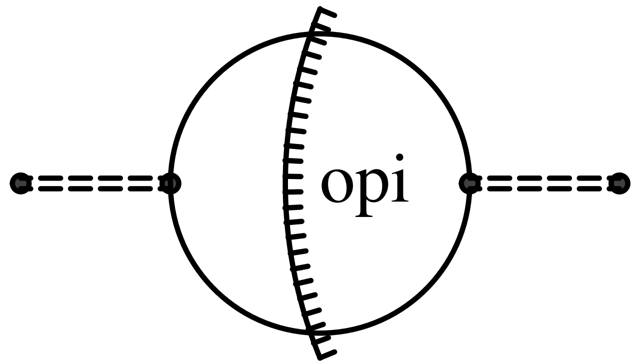
$$\frac{1}{p^2 + M^2 - F(p^2) - i\epsilon}$$

If $M > 2m$ then $i\epsilon$ may be left out, because $F(p^2)$ will have a non-zero imaginary part if $p^2 + M^2 = 0$

The imaginary part of $1/(p^2 + M^2 - F(p^2))$ is:

$$\frac{1}{p^2 + M^2 - F(p^2)}\ \mathrm{Im}(F)\ \frac{1}{p^2 + M^2 - F^*(p^2)}$$

This correponds to the cut diagrams

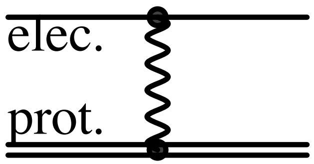


where the double dashed lines are the dressed B propagators. The cut propagator as occurring in the cut diagrams is replaced by a sum over states that the unstable particle can decay to.

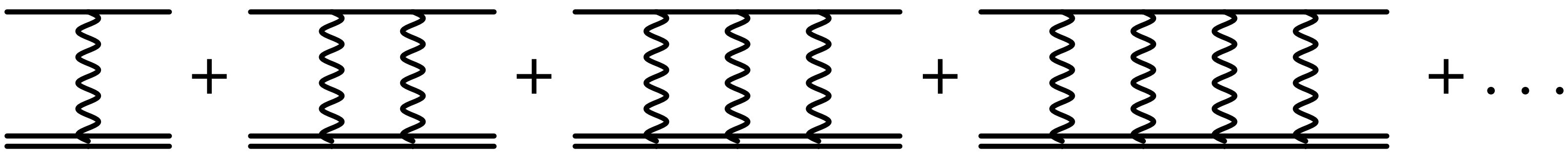
Bound States

Bound states such as the Hydrogen atom are essentially non-perturbative systems. If the coupling constant is zero there are no bound states, but for small but non-zero coupling constant there are. The situation is vaguely the opposite as with unstable particles. In the latter case particles disappear as in- or out-states; in the bound state situation new states with the bound particles as incoming or outgoing appear. Thus the Hydrogen atom, stable, can ooccur at minus or plus infinite time. How can this be understood in terms of diagrams ? A particle that can occur in this way is associated with a propagator that is singular for some value of the momentum ($p^2 + m^2 = 0$). How can such a propagotor come about? Again a partial summation may do the trick.

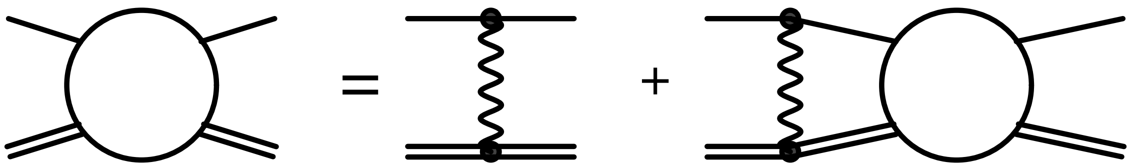
The starting point is the physical idea that an electron bound in an orbit around a proton remains in that orbit while constantly exchanging virtual photons taking care of the continuously changing momentum of he electron. Basic diagram:



This must be done many, many times. In short consider all "ladder" diagrams:

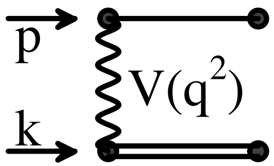


Let a blob denote this whole sum. Equation:



This equation is essentially the Schrödinger equation.

The essential ingredient: approximation low energy exchange. One step of the ladder:



$$V(q^2) = \frac{1}{q^2} \approx \frac{1}{\overrightarrow{q}^2} = V(\overrightarrow{q}^2)$$

The expression is (m, M = electron, proton mass):

$$C \int d_3q \; V(\overrightarrow{q}^2) \frac{4mM \;\; dq_0}{((p-q)^2 + m^2 - i\epsilon)((k+q)^2 + M^2 - i\epsilon)}$$

e-P restframe $\overrightarrow{k} = -\overrightarrow{p}$. Define $E_e = p_0 - m$, $E_P = k_0 - M$. Ignore q_0^2, E_e^2 and E_P^2. Further $\overrightarrow{Q} = \overrightarrow{p} - \overrightarrow{q}$

We will concentrate on the q_0 integration. The integrand becomes:

$$\frac{dq_0}{(2p_0q_0 + \overrightarrow{Q}^2 - 2mE_e - i\epsilon)(-2k_0q_0 + \overrightarrow{Q}^2 - 2MEp - i\epsilon)}$$

In terms of q_0 the first factor has a pole in the upper q_0 plane, the second in the lower plane. Close the contour with a half circle in the lower plane. The second factor has a pole there. Computing the residue one finds for the q_0 integral:

$$\frac{\pi i}{(p_0 + k_0)\overrightarrow{Q}^2 - 2k_0E_em - 2p_0E_PM - i\epsilon}$$

Setting now $p_0 \approx m, k_0 \approx M$ and $E =$ total e-P energy the result is now ($\mu =$reduced mass)

$$C \int d_3qV(\overrightarrow{q})\frac{1}{\frac{1}{2\mu}(\overrightarrow{p} - \overrightarrow{q})^2 - E - i\epsilon}$$

Shifting the integral $\overrightarrow{q} \rightarrow \overrightarrow{p} - \overrightarrow{q}$:

$$C \int d_3q \; \frac{1}{E - \frac{1}{2\mu}\overrightarrow{q}^2 + i\epsilon} \; V(\overrightarrow{p} - \overrightarrow{q})$$

The ladder equation for inital elecron momentum $\overrightarrow{p}$ and outgoing electron momentum $\overrightarrow{q}$:

$$F(\overrightarrow{q}, \overrightarrow{p}) = \delta_3(\overrightarrow{p} - \overrightarrow{q}) + C\int d_3k \frac{1}{E - \frac{1}{2\mu}\overrightarrow{q}^2 + i\epsilon} \times V(\overrightarrow{k} - \overrightarrow{q})F(\overrightarrow{k}, \overrightarrow{p})$$

This now includes the no-scattering part, which is the identity, $\delta_3(\overrightarrow{p} - \overrightarrow{q})$. Taking the fouriertransform with respect to $\overrightarrow{q}$ gives

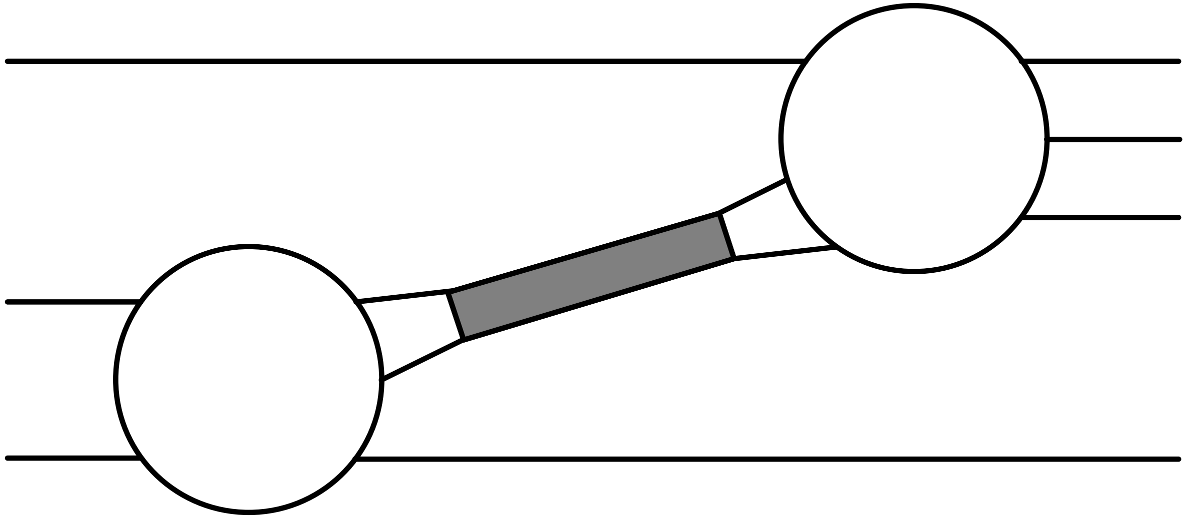
$$f(\overrightarrow{r}, \overrightarrow{p}) = e^{i\overrightarrow{p}\,\overrightarrow{r}} + \int d_3r' G(\overrightarrow{r}, \overrightarrow{r}')V(\overrightarrow{r}')f(\overrightarrow{r}', \overrightarrow{p})$$

This equation can be recognized as the Lippman-Schwinger equation, which is the Schrödinger equation applied to electron-proton scattering. It may be seen that for $E = \overrightarrow{p}^2/2\mu$ the function f is the solution of the equation

$$\left(\frac{1}{2\mu} + E\right) f(\overrightarrow{r}, \overrightarrow{p}) = V(\overrightarrow{r}) f(\overrightarrow{r}, \overrightarrow{p})$$

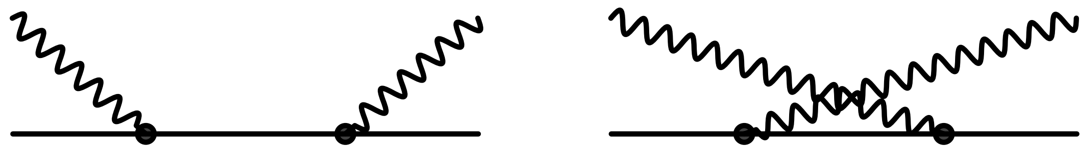
The bound state solutions of the Schrödinger equation will appear here, for off-mass shell electron and proton, as solutions for the scattering amplitude that have a pole in the complex energy plane, in fact behave around those bound state energies as $1/(E - E_n - i\epsilon)$, which is precisely as the propagtor for a particle at rest. Thus the bound states appear as propagating particles inside the S-matrix.

Example:



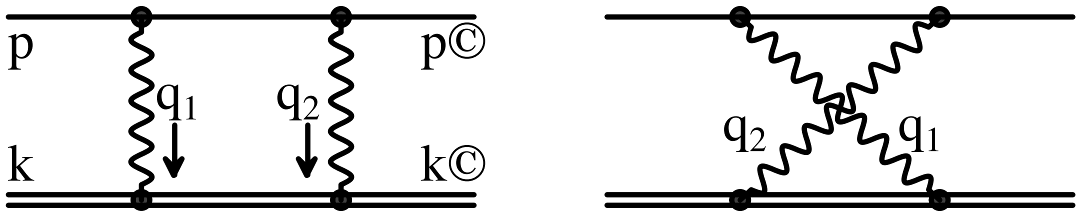
Bound states and gauge invariance

There is an important issue that must be discussed, namely gauge invariance. Consider photon-electron scattering. In fourth order there are two diagrams.



The second diagram is obtained by permuting the photon lines. General rule: to obtain a gauge invariant set one must include, for a given diagram, all diagrams that are obtained by permuting all photon lines.

Consider the two photon exchange contribution. To have a gauge invariant set the second diagram should be included.



The ladder approximation is not gauge invariant.
First diagram ($\Delta = p - p'$):

$$C \int d_4 q_1 d_4 q_2 \delta_4(q_1 + q_2 - \Delta) \delta_4(\Delta + k - k') V(q_1) V(q_2)$$

$$\frac{2m}{(p - q_1)^2 + m^2 - i\epsilon} \; \frac{2M}{(k + q_1)^2 + M^2 - i\epsilon}$$

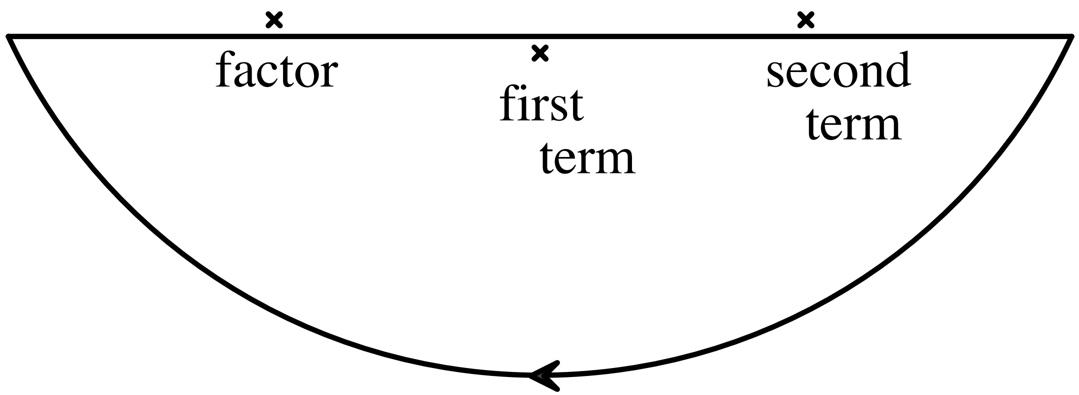
Second diagram: same except $k + q_1 \to k + q_2$.

Approx.: energy transfer $\ll$ mom. transfer.
$q_{20} = -q_{10}$, $E_e = p_0 - m$, $E_p = k_0 - M$.

$$\int dq_{10} \frac{2m}{2p_0 q_{10} + \overrightarrow{q}_1^2 - 2mE_e - i\epsilon}$$

$$\left[\frac{2M}{-2k_0 q_{10} + \overrightarrow{q}_1^2 - 2ME_p - i\epsilon} + \frac{2M}{2k_0 q_{10} + \overrightarrow{q}_2^2 - 2ME_p - i\epsilon} \right]$$

Consider q_{10} integral in complex plane.



Second term gives zero.

Try to include all possible crossed diagrams. Consider first the two photon exchange case. Proton restsystem, i.e. $E_p = 0$ and $k_0 = M$. Neglect $|\overrightarrow{q}|^2$. The terms between the square brackets become:

$$-\frac{1}{-q_{10} - i\epsilon} - \frac{1}{-q_{20} - i\epsilon}$$

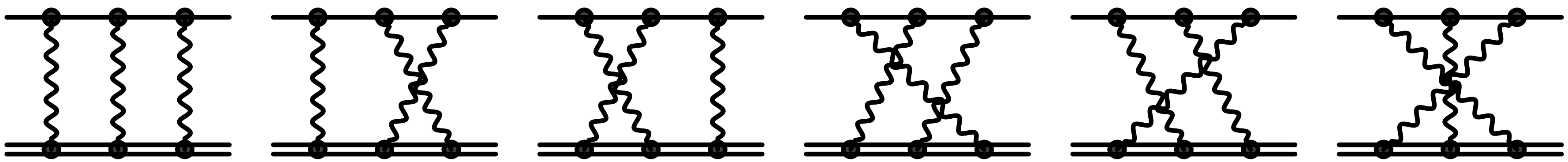
Use the well known equation

$$\frac{1}{x + i\epsilon} = P\left[\frac{1}{x}\right] - i\pi\delta(x)$$

where P denotes the principal value. With $q_{20} = -q_{10}$ the principal values are seen to cancel, and the two δ-functions add up to make $2i\pi\delta(q_{10})$ which is the same as obtained before, for the first term only.

Obviously this method is really quite dangerous. Trying naively to extend this to multiple photon exchange gives trouble. Three photon exchange:

$$\frac{\delta(q_{10}+q_{20}+q_{30})}{(q_{10}+i\epsilon)(q_{10}+q_{20}+i\epsilon)}+\frac{\delta(q_{10}+q_{20}+q_{30})}{(q_{10}+i\epsilon)(q_{10}+q_{30}+i\epsilon)}+...$$



There are six terms (the permutations over three objects). Naively evaluated gives six times $(i\pi)^2\delta()$. We will now prove the equation:

$$\Sigma_{\rm perm}\frac{\delta(\omega_1+\omega_2+\omega_3)}{(\omega_1-i\epsilon)(\omega_1+\omega_2-i\epsilon)}=(2i\pi)^2\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$$

Consider

$$\mathcal{M} = \Sigma_{\text{perm}} \int dt_1 dt_2 dt_3 e^{i\omega_1 t_1 + i\omega_2 t_2 + i\omega_3 t_3} \theta(t_3 - t_2)\theta(t_2 - t_1)$$

The term shown is zero unless $t_1 < t_2 < t_3$. Summing over all permulations of ω_1, ω_2, ω_3 is the same as summing over all permutations of the t. One obtains simply one for the product of the θ funtions (there is for any t_1, t_2 and t_3 one and only one non-zero θ expression). Therefore

$$\mathcal{M} = (2\pi)^3 \delta(\omega_1)\delta(\omega_2)\delta(\omega_3)$$

Using

$$\theta(z) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{e^{i\tau z}}{t - i\epsilon}$$

$$\mathcal{M} = \frac{1}{(2i\pi)^2} \Sigma \int d\tau_1 d\tau_2 dt_1 dt_2 dt_3 e^{i\omega_1 t_1 + i\omega_2 t_2 + i\omega_3 t_3}$$

$$e^{i\tau_2(t_3 - t_2)} \frac{1}{\tau_2 - i\epsilon} \quad e^{i\tau_1(t_2 - t_1)} \frac{1}{\tau_1 - i\epsilon}$$

$$= \frac{1}{(2i\pi)^2} \Sigma \int \frac{d\tau_1 d\tau_2}{(\tau_1 - i\epsilon)(\tau_2 - i\epsilon)}$$

$$(2\pi)^3 \delta(\omega_1 - \tau_1) \delta(\omega_2 - \tau_2 + \tau_1) \delta(\omega_3 + \tau_2)$$

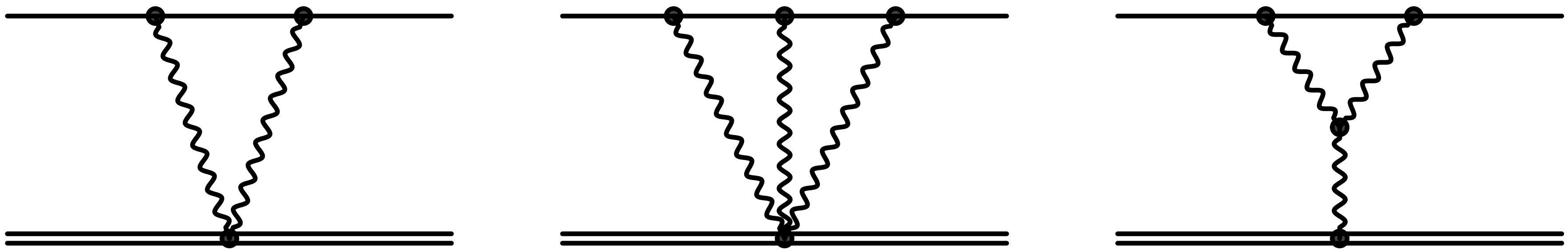
$$= \frac{(2\pi)^3}{(2i\pi)^2} \Sigma \frac{\delta(\omega_1 + \omega_2 + \omega_3)}{(\omega_1 - i\epsilon)(\omega_1 + \omega_2 - i\epsilon)}$$

Easily generalized to many photon exchange.

Bound states and gravitation

How about gravitation? In the first instance may be treated like for the electromagnetic case, i.e. summing ladder diagrams will give the gravitational bound states.

However, gauge invariance becomes a much more difficult issue. One must include many more diagrams in order to obtain a gauge invariant set. Here some examples:



This issue has not been discussed by anyone as far as I know.

Calculating one-loop diagrams

In doing actual calculations of one loop diagrams there are some essential tricks. Next to dimensional regularization and the Feynman trick for combining denominators:

$$\frac{1}{ab} = \int_0^1 \frac{1}{(ax + b(1-x))^2}$$

there is another very important trick. The point is that certain integrals can be more easily calculated if certain particles are massless (internal or external). Consider the product of two propagators:

$$\frac{1}{((q+p)^2 + m_1^2 - i\epsilon)((q+p+k)^2 + m_2^2 - i\epsilon)} = \frac{1}{AB}$$

Multiply numerator and denominator with $(1-\alpha)A+\alpha B$.

$$= \frac{1-\alpha}{B\left((1-\alpha)A+\alpha B\right)} + \frac{\alpha}{A\left((1-\alpha)A+\alpha B\right)}$$

The second denominator, $(1-\alpha)A+\alpha B$, is of the form $(q+l)^2+M^2-i\epsilon$ with $l=p+\alpha k$ and

$$M^2 = \alpha(1-\alpha)k^2+\alpha m_2^2+(1-\alpha)m_1^2.$$

By a suitable choice of α one can make $M^2=0$ (if k is spacelike) or $l^2=0$ (if either k or p are timelike).

To summarize our conventions consider the decay rate of a particle of momentum p into particles with momenta k_1, k_2, $\ldots$ (V = volume):

$$\Gamma = \int \frac{V d_3 k_1}{(2\pi)^3} \int \frac{V d_3 k_2}{(2\pi)^3} \cdots$$
$$P \frac{V}{(2\pi)^4} |\mathcal{S}|^2 \, \delta_4(p - k_1 - k_2 - \ldots) \, .$$

$\mathcal{S}$ is equal to the S-matrix element apart from the δ-function for overall energy-momentum conservation. The permutational factor P relates to identical particles in the final state: it contains a factor $1/n!$ for every set of n identical particles (bosons).

Concerning Lorentz invariance, $\mathcal{S}$ is of the form:

$$\mathcal{S} = \frac{\mathcal{M}}{\sqrt{2V\,p_0\,2V\,k_{10}\,2V\,k_{20}\ldots}}$$

where M is a Lorentz invariant expression. Squaring produces the Lorentz invariant measure $d_3k/2k_0$. The energy $k_0 = \sqrt{\vec{k}^{\,2} + m^2}$. Note:

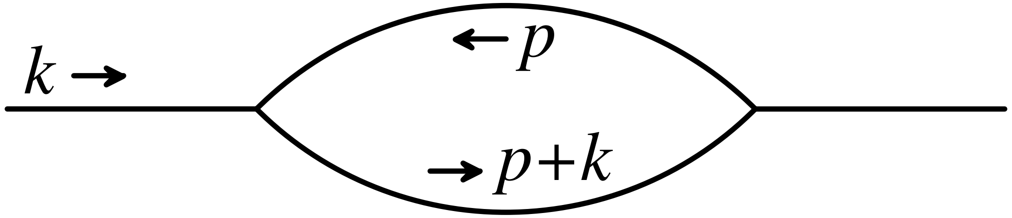
$$\int \frac{d_3k}{2k_0} = \int d_4k\,\theta(k_0)\,\delta(k^2 + m^2)\,.$$

k_0 is now integration variable, $d_4k = d_3k dk_0$, and $k^2 = \vec{k}^{\,2} - k_0^2$ is the four-dimensional dotproduct of k with itself. Using polar coordinates:

$$\int d_3k = \int_0^\infty k_\ell^2 dk_\ell \int_{-1}^{1} d\cos(\theta) \int_0^{2\pi} d\varphi\,,$$

where k_ℓ is the length of $\vec{k}$. A useful relation in this context is $k_\ell dk_\ell = k_0 dk_0$

Consider the simplest scalar one-loop diagram:



The corresponding expression is:

$$B_0(k, m, M) = \int \frac{d_n p}{(p^2 + m^2 - i\epsilon)((p+k)^2 + M^2 - i\epsilon)}$$

Step one is to use the Feynman parameter trick:

$$\frac{1}{ab} = \int_0^1 \frac{1}{[ax + b(1-x)]^2}$$

Result:

$$\int_0^1 dx \int d_n p \frac{1}{[p^2 + xm^2 + (1-x)M^2]^2}$$

This is where dimensional regularization comes in. The essential trick is to go to polar coordinates and do first the integration over all angles. The integrand will depend only on a finite number of angles. In this simple case there is no angular dependence in the integrand. Polar coordinates ($\mathbf{p}$ = length of $\mathbf{p}$):

$$\int d_n p = \int \mathbf{p}^{n-1} d\mathbf{p} \sin^{n-2}\theta_{n-1} d\theta_{n-1} \sin^{n-3}\theta_{n-2} d\theta_{n-2} \cdots$$

$$\int d\theta_1 = 2\pi; \text{ all others } \quad \int_0^\pi \sin^m \theta d\theta = \sqrt{\pi} \; \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}$$

The Γ function is the analytic continuation of the factorial: $\Gamma(z) = (z-1)!$ for integral z. The result is an expression that is meaningful for non-integer n.

So much for the essentials of dimensional integration. In our case we get finally:

$$B_0(k, m, M) = \int \frac{d_n p}{(p^2 + m^2 - i\epsilon)((p+k)^2 + M^2 - i\epsilon)}$$
$$= -i\Delta - i\pi^2 \int_0^1 dx \ln(\chi)$$

$$\chi = sx^2 + x(-s + M^2 - m^2) + m^2 - i\epsilon \qquad \text{with } s = -k^2$$

$$\Delta \equiv \frac{2\pi^2}{n-4} + \pi^2(\gamma + \ln(\pi)) - \pi^2 c$$

Suppose there is a momentum dependence in the numerator, for example:

$$B_{\mu\nu} = \int \frac{p_\mu p_\nu \quad d_n p}{(p^2 + m^2 - i\epsilon)((p+k)^2 + M^2 - i\epsilon)}$$

This expression contains only the vector k. It must therefore be of the form:

$$B_{\mu\nu} = B_{21}\delta_{\mu\nu} + B_{22}k_\mu k_\nu$$

Multiplying with $\delta_{\mu\nu}$:

$$nB_{21} + k^2 B_{22} = \int \frac{p^2 \; d_n p}{(p^2 + m^2 - i\epsilon)((p+k)^2 + M^2 - i\epsilon)}$$

$$= A(M^2) - m^2 B_0 \quad \text{with} \quad A(M^2) = \int \frac{d_n p}{p^2 + M^2}$$

Multiplying instead with $k_\mu k_\nu$ gives another equation for B_{12} and B_{22}. From these two equations B_{11} and B_{22} can be obtained.

Sofar about the two-point function. In practice one must deal also with three-point and four-point functions. The main difference is that the expressions become more complicated, involving not only logarithms but also Spence functions:

$$Sp(x) = -\int_0^1 dt \, \frac{\ln(1-xt)}{t}$$

Like other wellknown functions such as $\sin(x)$ this Spence function can be expanded in a series in x and worked out numerically. However, it is very hard to calculate it with great precision, i.e. in many decimals. Fortunately at the one-loop level no very high precision is needed for these Spence functions.

Concerning three and four-point functions with momenta in the numerator, similar techniques as demonstrated above for the two point function can be developed. It is a fortunate fact that one never deals with more than three equations with three unknows (like above the two equations with the two unknown functions B_{21} and B_{22}). However a real complication is that the resulting expressions are sums of terms with very nasty cancellations. As a consequence in numerical calculations individual terms must be calculated with sometimes monstruous precision (such as 60 decimals) in order to get a reasonably precise result.

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