Physics 406: Homework 10

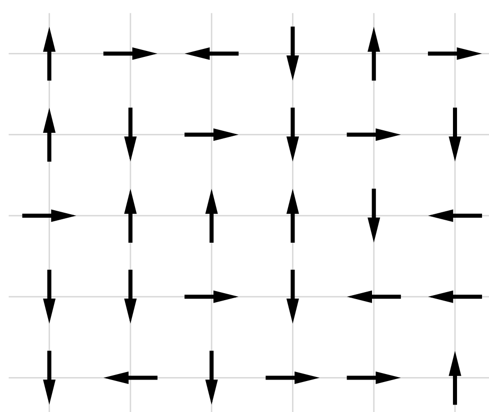
1. **Bose gas at low temperature:**

   (a) Write down an expression for the internal energy of a perfect Bose gas in terms of the density of states $n(\varepsilon)$ and the occupation number $f(\varepsilon)$. Assuming the gas is three-dimensional, put in the appropriate expressions for $n(\varepsilon)$ and $f(\varepsilon)$, being careful to include any terms for the Bose condensate of atoms in the ground state.

   (b) As we have seen, at low temperatures the activity is very close to 1. Change variables to $x = \varepsilon/\tau$, and so find an expression for the internal energy at low temperatures. Hence find the specific heat at low temperatures. How does this vary with temperature? Would it be possible to distinguish Bose from Fermi gases by the variation of the specific heat?

   (c) Find an expression for the entropy of the Bose gas at low temperature, by any means you like.

2. **Mean-field solution of a model magnet:** A model of a magnet consists of a grid of spins—elemental dipoles of unit magnetic moment—that can point in four different directions, north, south, east, and west, like this:



   Technically, this is called the **4-state clock model**.

   (a) Supposing all the spins to be independent, write down an expression for the mean magnetization m per spin in the vertical (north) direction if a magnetic field B is imposed in that direction.

   (b) Now suppose each spin feels the mean magnetic field of all the others, which is equal to $B = Jm$, where J is a positive constant. Show that the value of m must be a solution of the equation

$$m = \frac{\sinh(Jm/\tau)}{1+\cosh(Jm/\tau)}.$$

   (c) Hence show that this system goes through a continuous phase transition to a ferromagnetic state when the temperature falls below a critical value $\tau_c = \frac{1}{2}J$.

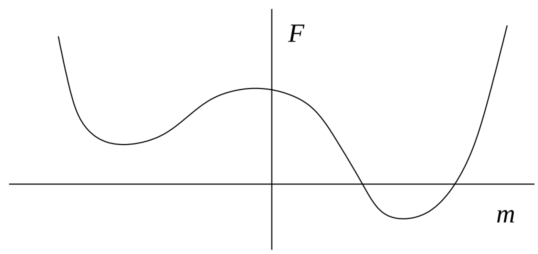
3. **Landau theory in a field:** Suppose we have a magnetic system in a magnetic field that gives the system a magnetic energy $-Bm$, where B is the applied field and m is the magnetization. Then the Landau free energy is

$$F = g_0 + g_1 m + g_2 m^2 + g_4 m^4 + \ldots = g_0 - Bm + \tfrac{1}{2}\alpha(\tau - \tau_c)m^2 + g_4 m^4 + \ldots$$

where g_0, g_4, and α are constants assumed independent of temperature and field. There is no term in m^3 because of the up/down symmetry of the system.

(a) Find the temperature at which the (second-order) phase transition to a spontaneously symmetry broken state occurs in zero magnetic field.

(b) Now suppose that $B > 0$. Then the free energy looks something like this:



Assuming that we are at $\tau < \tau_c$, find the equation satisfied by the two minima of the free energy. Which of the two minima corresponds to the equilibrium state?

(c) If the field is slowly reduced (made more negative), while keeping the temperature fixed, the relative heights of the two minima vary. However, once the system is in one minimum, it cannot switch to the other if there is a hump in the between the two minima—it would need to increase its energy to do so. What negative value must the field attain go before we flip over to the other minimum? And how far positive must it go after that to flip us back again?

(d) Hence draw a sketch of the behavior of m as a function of B.