# Physics 406: Homework 4

1. **Two spin systems in contact:** A result which we will use a lot in this course is Sterling's approximation for the logarithm of a factorial, which says that

$$\ln n! \simeq n \ln n - n,$$

where the approximation becomes better and better as $n \to \infty$. We will see how to prove this result in a later lecture. For the moment we just assume it. (If you want to see the proof, it's given in Appendix A of Kittel and Kroemer.)

Now consider the spin system we looked at, composed of $N$ "spins" that each point either up or down.

   (a) Write down an expression for the multiplicity $g(N,s)$ of states that have a given value of the spin excess parameter $s$, defined by $2s = N_\uparrow - N_\downarrow$. Hence write down an expression for the logarithm of the multiplicity as a function of $s$.

   (b) Apply Sterling's approximation and show that for large $N$

$$\ln g = N \ln N - (\tfrac{1}{2}N + s)\ln(\tfrac{1}{2}N + s) - (\tfrac{1}{2}N - s)\ln(\tfrac{1}{2}N - s). \tag{1}$$

   (c) Noting that $\ln(\frac{1}{2}N + s) = \ln(\frac{1}{2}N) + \ln(1 + 2s/N)$, expand to second order in $2s/N$ and hence show that

$$g(N,s) \simeq 2^N \mathrm{e}^{-2s^2/N}. \tag{2}$$

   This is a Gaussian or normal distribution: the binomial distribution becomes a Gaussian distribution for large $N$.

   (d) Now suppose we have two identical systems of $N$ spins. Using Eq. (2), write down an expression for the total multiplicity of both systems together as a function of the spin excesses $s_1$ and $s_2$ of the two systems, assuming for the moment that the two systems are not in contact with one another. (To make the calculations easier, you can assume $N$ is even.) Eliminate $s_2$ in favor of the total spin excess $s = s_1 + s_2$ and hence show that the distribution of possible values of $s_1$ is Gaussian.

   (e) Now suppose that both systems are in a magnetic field of intensity $B$, so that they have energies $U_1 = 2mBs_1$ and $U_2 = 2mBs_2$, where $m$ is the dipole moment of each spin. If the two systems are now in contact with one another, so that energy can flow between them, then $s_1$ and $s_2$ are no longer fixed, but the total energy must be conserved, so $s = s_1 + s_2$ is constant. What is the most likely value of $s_1$?

2. **Entropy of a set of harmonic oscillators:** Consider the ensemble that we discussed earlier of $N$ quantum harmonic oscillators. Each can have energy $\varepsilon_s = s\hbar\omega$, where $\hbar$ and $\omega$ are constants and $s$ is a non-negative integer.

   (a) Write down the expression for the multiplicity $g(N,n)$ of states of the entire ensemble that have total internal energy $U = n\hbar\omega$, in a form involving factorials and hence write down the dimensionless entropy $\sigma$ of the system.

   (b) When $N$ is large we can, to a good approximation, replace $N-1$ by $N$. Apply Sterling's approximation and derive an approximate expression for $\sigma$ for large $N$.

(c) Recalling the definition of the temperature $\tau$ (in energy units) $\tau^{-1} = \partial\sigma/\partial U|_N$, differentiate to get an expression for $\tau$ in terms of the internal energy. (You have to consider $n$ to be a continuous variable to do this calculation, which is strictly speaking not correct—it is an integer. Later in the course we'll see a better derivation of this result that doesn't require us to do this kludge.)

(d) Rearrange to show that

$$U = \frac{N\hbar\omega}{\exp(\hbar\omega/\tau) - 1}.$$

This is the internal energy of a set of $N$ harmonic oscillators and, as we will show, is also the correct expression for the energy of a set of bosons (e.g., photons) in a quantum gas. From this expression we will later derive the famous black-body radiation spectrum of Rayleigh and Planck.

(e) What is the heat capacity of the system?

3. **Partition function of a simple system:** Suppose a simple system has states with three energies, $-\varepsilon$, 0, and $+\varepsilon$. The multiplicities of the states are $g(-\varepsilon) = 1$, $g(0) = 2$, and $g(\varepsilon) = 1$. The system is put in contact with a thermal reservoir at dimensionless temperature $\tau$ and allowed to come to equilibrium.

   (a) Calculate the partition function $Z$ of the system.

   (b) Calculate the average internal energy of the system as a function of temperature.

   (c) Show that the heat capacity is

$$C = \frac{\varepsilon^2}{\tau^2[1 + \cosh(\varepsilon/\tau)]}.$$