Physics 406: Homework 5

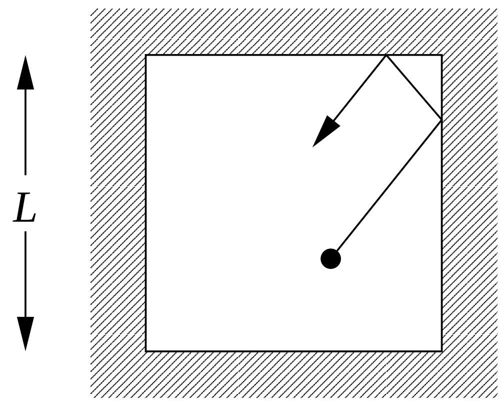
1. **Entropy of the two-state system:** In class we looked at a system with two states of energies 0 and ε, in equilibrium with a thermal reservoir at temperature τ.

   (a) For this system, calculate the internal energy U and the free energy F.

   (b) Thus calculate the entropy of the system σ.

   (c) What is the value of the entropy at large temperatures? What is the simple physical reason for this value?

2. **The perfect gas in Flatland:** In the mythical two-dimensional universe called Flatland, there exists a square box, of length L on each side:



   The single atom bouncing around in this box is described by the Schrödinger wavefunction

$$\psi(x,y) = \psi_0 \sin\left(\frac{\pi}{L} n_x x\right) \sin\left(\frac{\pi}{L} n_y y\right),$$

   at position x, y, where $n_x, n_y = 1, 2, 3\ldots$ and ψ_0 is a normalizing constant.

   (a) Substitute this wavefunction into the Schrödinger equation $\varepsilon\psi = (\hbar^2/2m)\nabla^2\psi$ to get the energy ε of the particle in terms of its mass m and Planck's constant.

   (b) Hence write an expression for the partition function Z_1 of the single particle. Using an integral approximation, find Z_1 as a function of the area A of the box. Using this expression, write down the partition function for the two-dimensional perfect gas, i.e., a gas of N non-interacting identical particles.

   (c) What is the heat capacity of this gas?

   (d) What is the equation of state for the two-dimensional perfect gas, and what is the value of the gas constant for one mol?

3. **Integral approximations:** Using integral approximations, find the approximate values of the sums:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+e^{\alpha n}}$, (b) $\displaystyle\sum_{n=1}^{N} \ln n$, (c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\beta+n)^3}$,

   where α and β are constants. Hint: notice the limits of the sums. (You can work out the integrals quite easily, but you can also look them up in the tables if you prefer.)

4. **Fluctuations in the energy:** A system has many states denoted by $i = 1, 2, 3 \ldots$ with energies ε_i. It is in equilibrium with a thermal reservoir at temperature τ, which means that it hops from one state to another over time. Thus there will be fluctuations in the energy of the system—small amounts of energy will enter and leave the system from the reservoir. Just as we calculated the width of distributions previously by calculating their standard deviation, we can calculate the width of the energy distribution, i.e., the size of the energy fluctuations, by calculating the standard deviation of energy.

   (a) Write down the partition function for the system, and differentiate it to show that the internal energy of the system is

   $$U = \langle \varepsilon \rangle = \tau^2 \frac{\partial \ln Z}{\partial \tau}.$$

   (b) Now show that the mean *squared* energy of the system is

   $$\langle \varepsilon^2 \rangle = \frac{\tau^2}{Z} \frac{\partial}{\partial \tau} \left(\tau^2 \frac{\partial Z}{\partial \tau} \right).$$

   (c) Hence show that the standard deviation $\sigma_U = \sqrt{\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2}$ of the energy is given by

   $$\sigma_U = \tau \sqrt{\frac{\partial U}{\partial \tau}}.$$

   (d) Rewrite this in terms of the heat capacity of the system. Note thus that the fluctuations in the energy—a microscopic quantity—are directly related to the heat capacity—a macroscopic quantity. By measuring one, we can measure the other. This result is often used in computer simulations of thermal systems. You measure the fluctuations of the energy and so calculate what the heat capacity must be.