Physics 406: Homework 6

1. **Number of photons in the universe:**

   (a) Calculate the total number of photons of all frequencies in a cavity of volume V at equilibrium at temperature T. You will need to know that

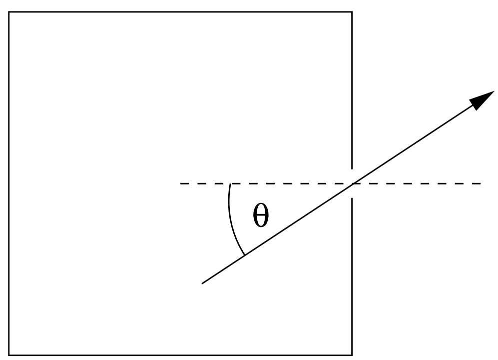
   $$\int_0^\infty \frac{x^2}{e^x - 1}\,dx = 2\zeta(3) = 1.202\ldots,$$

   where $\zeta(x)$ is the Riemann ζ-function.

   (b) The diameter of the universe is about 20 billion light years. About how many photons are there in the universe?

   (c) Supposing that no photons enter or leave the universe as it expands, how are the temperature and volume of the universe related?

2. **The Stefan-Boltzmann constant:** We have argued that a small hole in the side of a box at temperature T radiates as a black body at that temperature. Suppose the hole has area A and consider the radiation that escapes from the box through the hole:



   We have shown that the energy density in the box is

   $$u = \frac{\pi^2 k_B^4 T^4}{15\hbar^3 c^3},$$

   where c is the speed of light. Consider the radiation incident on the hole from a direction that makes an angle θ with the normal to the hole, and arrives within a solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$ of that direction.

   (a) What is the energy in a small volume element $r^2\,dr\,d\Omega$, where r is measured from the hole? And what fraction of this actually hits the hole, rather than going in some other direction? Thus integrate over r from 0 to c to find the amount of energy transported through the hole per unit time from the given solid angle $d\Omega$.

   (b) Integrate over θ and ϕ to get the total radiation leaving the hole per unit time. (Hint: be careful about the limits of the integration.)

   (c) The amount of radiation J per unit area given off by a black body at temperature T is given by the Stefan-Boltzmann law:

   $$J = \sigma_B T^4,$$

   where σ_B is the Stefan-Boltzmann constant. Show that the Stefan-Boltzmann constant has the value $5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.

3. **The maximum of the thermal spectrum:** We have shown that the radiation energy per unit volume in a cavity at temperature τ between frequencies ω and $\omega + \mathrm{d}\omega$ is

$$u(\omega)\,\mathrm{d}\omega = \left[\frac{\hbar}{\pi^2 c^3}\right] \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}\,\mathrm{d}\omega.$$

   (a) At what frequency ω_{max} is the radiation a maximum? You will need the result that the solution to the equation $3\mathrm{e}^{-x} + x - 3 = 0$ is $x = 2.82\ldots$

   (b) The wavelength of electromagnetic radiation is related to its angular frequency by $\lambda\omega = 2\pi c$. Find the radiation energy between wavelengths λ and $\lambda + \mathrm{d}\lambda$.

   (c) Hence find the wavelength λ_{max} at which the radiation is a maximum. You will need the result that the solution to the equation $5\mathrm{e}^{-y} + y - 5 = 0$ is $y = 4.96\ldots$

   (d) One might expect that $\lambda_{\mathrm{max}}\omega_{\mathrm{max}} = 2\pi c$. Why is this not the case?

4. **Radiation pressure:**

   (a) Show that the free energy F of the radiation in a cavity of volume V at temperature T is

$$F = \frac{V\tau^4}{\pi^2\hbar^3 c^3}\int_0^\infty x^2 \ln(1 - \mathrm{e}^{-x})\,\mathrm{d}x = -\frac{V\tau^4\pi^2}{45\hbar^3 c^3}.$$

   (Hint: look at the midterm if you don't know how to do this.)

   (b) Hence find the pressure exerted by the radiation on the walls of the box. This pressure is called **radiation pressure**.