Physics 406: Homework 7

1. **Phonon speci£c heat above the Debye temperature:** We have shown that the internal energy U of the phonons in a solid is

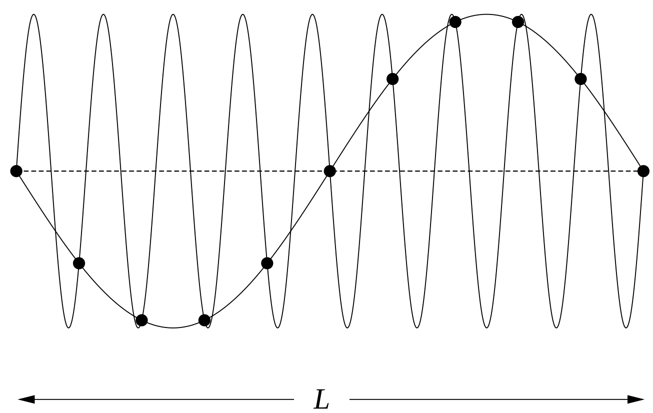
$$U = \frac{9N\tau^4}{\theta^3} \int_0^{x_{\max}} \frac{x^3}{e^x - 1}\, dx,$$

where $x_{\max} = \theta/\tau$ with the Debye temperature θ being given by

$$\theta = \sqrt[3]{6\pi^2\hbar^3 v^3 \rho}, \qquad \rho = \frac{N}{V}.$$

When $\tau \gg \theta$ show by expanding the integrand above that the value of the heat capacity is $C = 3N$ to leading order *and* to second order—i.e., that including the next order term in the expansion makes no difference to the value of the heat capacity at high temperature.

2. **Aliasing:** Here's a picture representing the aliasing phenomenon that's much better than the one I drew in class:



It depicts a one-dimensional version of the aliasing problem. The dots are atoms. The solid lines represent two possible waves, both of which correspond to the same positions of the atoms.

Suppose the length of the system in atom spacings is L, as shown. In the picture $L = 10$. Let the atoms be numbered $k = 0 \ldots L$. Then the displacement y_k of the kth atom is

$$y_k = \sin(2\pi k/\lambda_h),$$

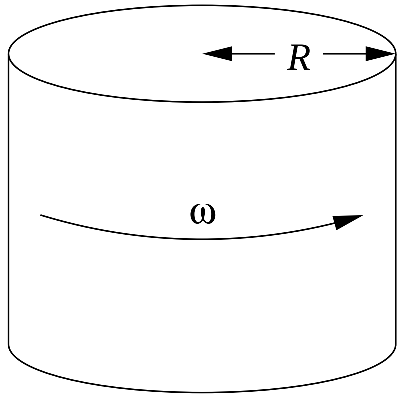
where λ_h is the wavelength of the high-frequency wave, which is related to the frequency by $\lambda_h \omega_h = 2\pi v$, with v being the speed of sound.

(a) Show that if $\lambda_h = 2L/n$, where n is any integer, then the wave is zero at both ends of the system. How many modes are there between frequencies ω and $\omega + d\omega$?

(b) Show that another similar wave with the longer wavelength $\lambda_l = 2L/(2L - n)$ gives the same displacements of the atoms (except for a minus sign).

(c) Hence what is the Debye frequency $\omega_{\max}$ for this one dimensional solid (i.e., the maximum frequency of a unique wave in the system)?

3. **Centrifuge:** A centrifuge consists of a circular cylinder of radius R, spinning about its axis with angular velocity ω:



It contains an ideal gas of atoms that have mass m in equilibrium at temperature τ. Assuming all the gas in the cylinder spins with the cylinder:

(a) Write down an expression for the total chemical potential at radius r in the gas in terms of the density $\rho(r)$, including the internal (perfect gas) part and the external (kinetic energy) part. Be careful about the sign of the kinetic energy part; you need to think about the problem in the rotating frame of reference of the centrifuge, in which frame there is a force on the particles towards the outside of the circle.

(b) Hence derive an expression for the density in terms of its value $\rho(0)$ at the axis of the cylinder.

4. **Gibbs distribution for a simple system:** Suppose we have a simple system that has three possible states. Either it can have no particles in it, in which case it has energy 0, or it can have one particle with either energy 0 or energy ε.

(a) Write down an expression for the grand partition function $\mathcal{Z}$.

(b) Find the average number of particles in the system as a function of τ and the chemical potential.

(c) Find the average internal energy of the system.