Physics 406: Homework 9

1. **Internal energy in the grand ensemble:** We have seen that the grand potential Ω is given by

$$\Omega = -\tau \log \mathcal{Z} = U - \tau\sigma - \mu N.$$

   (a) Using the expressions for σ and N in terms of Ω show that the internal energy U is given by

$$\frac{U}{\tau} = \tau \frac{\partial \log \mathcal{Z}}{\partial \tau}\bigg|_{\mu} + \mu \frac{\partial \log \mathcal{Z}}{\partial \mu}\bigg|_{\tau}.$$

   (b) Prove that for any function $f(\tau,\lambda)$, where $\lambda = e^{\mu/\tau}$ is the activity of the system

$$\frac{\partial f}{\partial \tau}\bigg|_{\lambda} = \frac{\partial f}{\partial \tau}\bigg|_{\mu} + \frac{\partial f}{\partial \mu}\bigg|_{\tau}\frac{\partial \mu}{\partial \tau}\bigg|_{\lambda} = \frac{\partial f}{\partial \tau}\bigg|_{\mu} + \frac{\partial f}{\partial \mu}\bigg|_{\tau}\frac{\mu}{\tau}.$$

   (c) Hence show that U can also be written

$$U = \tau^2 \frac{\partial \log \mathcal{Z}}{\partial \tau}\bigg|_{\lambda}.$$

   This is the equivalent of the familiar expression $U = \tau^2(\partial \log Z/\partial\tau)$ for the canonical ensemble.

   (d) Use this result to show that the internal energy of the classical perfect gas is $U = \frac{3}{2}N\tau$. You will need the result for the value of N in a perfect gas that we derived in class.

2. **Variation of the chemical potential with temperature:** We've seen that the probability of a particle in a Fermi gas having energy equal to the chemical potential μ is $\frac{1}{2}$. But remember that μ can depend on temperature, so that the point where $f(\varepsilon) = \frac{1}{2}$ is not always in the same place. Here we calculate how μ varies for the two-dimensional Fermi gas. You can do the same calculation in the three-dimensional case, but the integrals are harder. (In three dimensions they involve the polylogarithm function.)

   (a) The energy states of a particle in a two dimensional box of size $L \times L$ satisfy

$$\varepsilon = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2),$$

   where m is the mass of the particle. Show that the density of states, i.e., the number of states per unit interval of energy, is

$$n(\varepsilon) = \frac{Am}{\pi\hbar^2},$$

   independent of energy, where $A = L^2$ is the area of the box.

   (b) Write an expression for the number N of particles (assumed to be fermions) in the box in terms of the density of states and the Fermi function $f(\varepsilon)$. Plugging in the expressions for $n(\varepsilon)$ and $f(\varepsilon)$, show that

$$N = \frac{Am}{\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/\tau} + 1},$$

and hence that the activity is given by

$$\lambda = \exp\left(\frac{\pi\hbar^2\rho}{m\tau}\right) - 1,$$

where $\rho = N/A$ is the density of the gas. You will need the result that

$$\int_0^\infty \frac{\mathrm{d}x}{a\mathrm{e}^x + 1} = \log(1 + a^{-1}).$$

(c) What value does this give for μ as $\tau \to 0$? And as temperature rises from absolute zero, which way does the chemical potential move—up or down?

3. **^{3}He as a Fermi gas:** Liquid ^{3}He, the light isotope of helium, has spin $\frac{1}{2}$ and so is a Fermion.

(a) Treating ^{3}He as a perfect gas (which is only approximately right) and noting that its density is $0.081\,\mathrm{g\,cm^{-3}}$, calculate the Fermi energy ε_F. What is the corresponding Fermi temperature (i.e., the Fermi energy expressed in temperature units)? This gives us a rough measure of how cold the ^{3}He has to be before we start seeing quantum effects.

(b) Calculate the heat capacity of ^{3}He at temperatures much less than the Fermi temperature. The real heat capacity is measured to be $2.89NT$. How close is your estimate to the real figure?