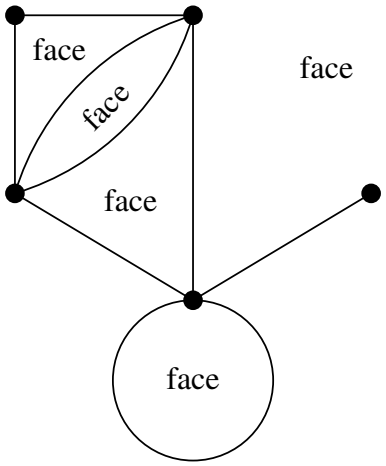
Complex Systems 535/Physics 508: Homework 2

1. **Mean degree of a planar graph:** Consider a connected planar graph with n vertices and m edges. Let f by the number of "faces" of the graph, i.e., areas bounded by edges when the graph is drawn in planar form. The "outside" of the graph, the area extending to infinity on all sides, is also considered a face. The graph can have self-loops and multiedges.



   (i) Write down the values of n, m, and f for a graph with a single vertex and a single self-loop.

   (ii) How do n, m, and f change when we add a single vertex to the graph along with a single edge attaching it to another vertex?

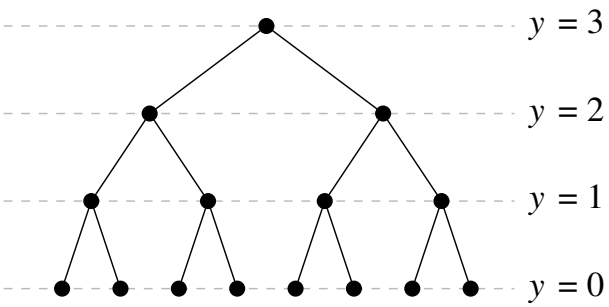
   (iii) How do n, m, and f change when we add a single edge between two extant vertices (or a self-loop attached to just one vertex), in such a way as to maintain the planarity of the graph?

   (iv) Hence by induction prove a general relation between n, m, and f for all connected planar graphs.

   (v) Now suppose our graph is simple (i.e., contains no multiedges) and loop-free. Show that the mean degree z of such a graph is strictly less than six.

   The mean degree of intersections in the US interstate road network (which is very nearly planar) is 2.86.

2. **Betweenness centrality for a tree:** Consider a regular binary tree, that is, a tree with constant branching ratio 2 and hence constant degree 3 for all vertices other than the root and the leaves:



   Let y be the height of a vertex up the tree with the leaves having height 0. Let Y be the height of the root.

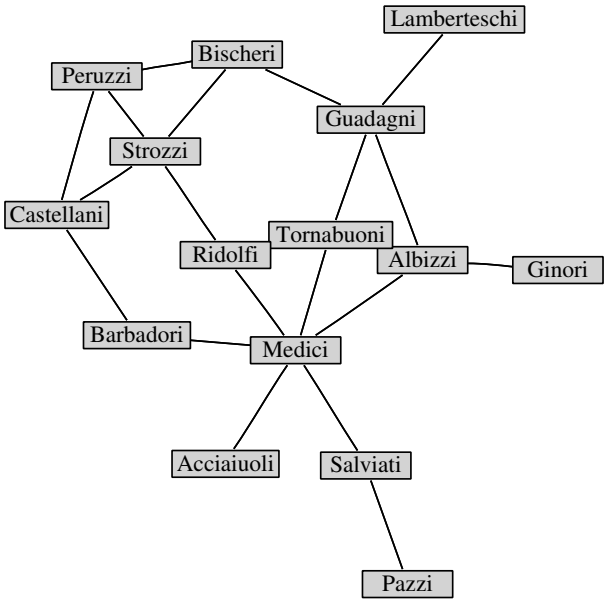
(i) How many vertices are there in the tree in all as a function of Y?

(ii) Consider a vertex A at height y. How many vertices are directly below A in the tree, including A itself? (Here "below" means that you would have to pass through A to get to them from the root.) How many geodesic paths between pairs of those vertices does A lie on, again including paths to and from itself?

(iii) Hence what is the betweenness centrality of the root node as a function of Y? And as a function of the size n of the network?

The root node has the highest betweenness in this network and increases about as fast as possible, given that there are $O(n^2)$ total geodesic paths on a tree of n vertices. Similar results are found for many non-tree-like networks as well. This is why betweenness is such a good discriminator for centrality—there is a big difference between the highest and lowest scores in the network.

3. **Eigenvector centrality:** Here is a famous network from the field of social network analysis:



In this network the vertices are influential families in 15th century Florence, and the edges represent intermarriage between families (Padgett and Ansell (1993) *American Journal of Sociology* **98**, 1259–1319).

(i) Which vertex has the highest degree centrality? And which the second highest?

(ii) Using your favorite calculator software (e.g., Matlab or Mathematica) find which vertex has the largest eigenvector centrality. Which comes second?

(iii) Give a (one sentence) explanation of the apparent conflict between the answers to (i) and (ii).

The Medici dominated Florentine trade for most of the latter part of the 15th century. It is believed that their dominance arose primarily through skillful manipulation of their network of social contacts.