Complex Systems 535/Physics 508: Homework 7

1. **Generating functions:**

   (i) The Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8... The generating function for the Fibonacci numbers is therefore $f(z) = z + z^2 + 2z^3 + 3z^4 + \ldots$ The Fibonacci numbers have the definitive property that each is the sum of the previous two. Show that $f(z) = z/(1 - z - z^2)$.

   (ii) Consider the binomial distribution of an integer variable k:

   $$B_{nk}(p,q) = \binom{n}{k} p^k q^{n-k},$$

   where $q = 1 - p$. Derive a closed-form expression for the generating function $g(z) = \sum_{k=0}^{n} B_{nk} z^k$. Hence find the mean, mean-square, and variance of the binomial distribution.

   (iii) A sequence of numbers a_k with $k = 1, 2, 3, \ldots$ satisfies the recurrence

   $$a_k = \begin{cases} 1 & \text{for } k = 1, \\ \sum_{j=1}^{k-1} a_j a_{k-j} & \text{for } k > 1. \end{cases}$$

   Show that the generating function $h(z) = \sum_{k=1}^{\infty} a_k z^k = \frac{1}{2}(1 - \sqrt{1 - 4z})$.

2. **Component sizes in the Bernoulli random graph:** Consider the Bernoulli random graph G_{np}.

   (i) Write down an expression satisfied by the generating function $H(x)$ for the size of the component to which a randomly chosen vertex belongs, in terms of the mean degree z of the graph.

   (ii) Hence give an expression for the expected number of components of size 2 on the graph in the limit of large n.

3. **Random graph with low degrees:** Suppose a generalized random graph has vertices with degrees 1, 2, and 3 only, with probabilities p_1, p_2, and p_3.

   (i) Write down the generating functions $G_0(x)$ and $G_1(x)$ for the degree distribution and the excess degree distribution.

   (ii) Hence, or otherwise, find an expression involving p_1, p_2, and p_3 that must be satisfied if there is to be a giant component in the graph.

   (iii) Derive a closed-form expression for the generating function $H_1(x)$ of the numbers of vertices reachable by following a randomly chosen edge.

4. **The Internet:** The Internet is found to have a power-law degree distribution $p_k \sim k^{-\alpha}$, with $\alpha \simeq 2.5$.

   (i) Make a mathematical model of the Internet using the configuration model with this degree distribution. Write down the fundamental generating functions G_0 and G_1.

   (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they can actually send data over the network to each other).