Complex Systems 899: Homework 3

1. **An example linear system:** Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

   (a) Show that the characteristic polynomial for the system is $\lambda^2 - 5\lambda + 6$ and so find the eigenvalues and eigenvectors.

   (b) Derive the general solution for the system.

   (c) Classify the fixed point at the origin.

   (d) Solve the system with the initial condition $x = 3$, $y = 4$.

2. **Phase portraits:** Plot the phase portrait and classify the fixed point at the origin for the following systems of equations:

   (a) $\dot{x} = y$, $\dot{y} = -2x - 3y$

   (b) $\dot{x} = 4x - 3y$, $\dot{y} = 8x - 6y$

3. **Trickier system:** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}.$$

   Solve the dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$ and sketch the phase portrait.

4. **Steve Strogatz' messed-up couple:** In class we considered the case of Romeo and Juliet, whose tragic love affair is described by a pair of unromantic differential equations in which $R(t)$ is Romeo's love (or hate) for Juliet at time t and $J(t)$ is Juliet's for Romeo. Find and describe what happens in each of the following cases, if $a, b > 0$:

   (a) Do opposites attract? $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$.

   (b) What if they have everything in common? $\dot{R} = aR + bJ$, $\dot{J} = bR + aJ$. Should they expect boredom or bliss?

   (c) Nothing can ever change the way Romeo feels: $\dot{R} = 0$, $\dot{J} = aR + bJ$. Does Juliet end up loving him or hating him?

5. **Linearization:** For each of the following systems, find the fixed points, classify them, sketch the flows around them, and then try to fill in the rest of the phase portrait:

   (a) $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

   (b) $\dot{x} = \sin y$, $\dot{y} = \cos x$