Complex Systems 899: Homework 4

This is the last of the continuous dynamical systems. After this it's discrete dynamics, random processes, game theory, and other fun stuff.

1. **An example of a 2D nonlinear system:** Consider the system defined by the equations $\dot{x} = y - 2x$, $\dot{y} = \mu + x^2 - y$, where μ is a parameter.
   (a) Solve for the positions of the fixed point(s).
   (b) Classify the bifurcation(s) that occur as μ varies.
   (c) Give a sketch of the phase portrait.

2. **Hopf bifurcation in a predator-prey model:** Let x, y be proportional to the populations of a predator and a prey in an ecosystem. The equations

$$\dot{x} = x^2(1-x) - xy, \qquad \dot{y} = xy - ay,$$

   have been proposed as describing the dynamics, with $a > 0$.
   (a) Give a brief interpretation of the equations in population dynamics terms. You can start off by saying which of x, y is the predator population and which is the prey.
   (b) Show that the fixed points are at $(0,0)$, $(1,0)$, and $(a, a-a^2)$ and classify the behavior at those points to linear order for values of a close to 1.
   (c) Demonstrate that the predators always go extinct if $a > 1$.
   (d) Show that a Hopf bifurcation occurs at $a_c = \frac{1}{2}$.
   (e) Estimate the frequency of the population oscillations near the bifurcation in terms of the parameter a.

3. **Hopf bifurcation in the Lorenz equations:** We have seen that there are three fixed points in the Lorenz equations for $r > 1$: the origin and the two points we called C^+ and C^-.
   (a) Find the Jacobian at the fixed points C^+ and C^- and show that its characteristic equation, which is now cubic because this is a 3D system, is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0.$$

   (b) A Hopf bifurcation occurs when eigenvalues cross the imaginary line—i.e., when they are pure imaginary. Look for purely imaginary solutions $\lambda = i\omega$ to the equation above (with ω real) and hence show that there is a Hopf bifurcation when

$$r = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

(c) You'll need to assume $\sigma > b - 1$. Why?

(d) Finally, find the third eigenvalue.

4. **The tent map:** We have seen that the Lorenz equations can be understood in terms of a *discrete* dynamical system, the Lorenz map. In fact a lot of aspects of chaotic systems can be understood from the study of such systems, and we'll spend a lecture or two shortly looking in detail at one famous example, the logistic map. Another much studied case is the "tent map," which is similar in shape and behavior to the Lorenz map, and can shed some light on the behavior of the Lorenz system. The tent map is a discrete iteration defined by

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \le x_n \le \frac{1}{2}, \\ 2 - 2x_n & \text{for } \frac{1}{2} \le x_n \le 1. \end{cases}$$

(a) Why is it called the "tent map"?

(b) Find all the fixed points and classify their stability.

(c) Show that the map has a period-2 orbit. Is it stable or unstable?

(d) Can you find any period-3 orbits? How about period-4? Are they stable or unstable?