Complex Systems 899: Homework 5

1. **The cubic map:** Consider the map $x_{n+1} = f(x_n)$ with $f(x) = rx - x^3$.

   (a) Find the fixed points. For what values of r do they exist? For what values are they stable?

   (b) To find the 2-cycles of the map, define $p = f(q)$ and $q = f(p)$. Then show that p and q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$ and hence find all the 2-cycles.

2. **Similarity transformations:** Consider a discrete map $x \leftarrow f(x)$ for some $f(x)$. For some function $h(x)$ define $g(x) = h^{-1}(f(h(x)))$, where $h^{-1}(x)$ is the functional inverse of $h(x)$.

   (a) If x^* is a fixed point of $g(x)$ show that $h(x^*)$ is a fixed point of $f(x)$.

   (b) If x_p belongs to a cycle of length p in $x \leftarrow g(x)$, show that $h(x_p)$ belongs to a cycle of length p in $x \leftarrow f(x)$.

   (c) If $f(x)$ is the logistic map for $r = 4$ and $h(x) = \sin^2 \pi x$, show that $g(x)$ is the tent map.

   (d) Hence find exact expressions for values of x that lie on a 2-cycle and two different 3-cycles of the logistic map at $r = 4$. (You'll need results from Homework 4 to complete this problem.)

3. **The shift map:** Let $\lfloor x \rfloor$ be the "floor" function—the largest integer not greater than x. Now consider the map $x_{n+1} = f(x_n)$ on the unit interval $0 \le x < 1$ where $f(x) = bx - \lfloor bx \rfloor$ for b integer, i.e., $f(x)$ is the fractional part of bx, the part after the decimal point. This is called the *shift map*.

   (a) Sketch a graph of $f(x)$ for some reasonable value of b.

   (b) Find all the fixed points for general b.

   (c) Show that the shift map has at least one periodic orbit of every period, but that all periodic orbits are unstable.

   (d) Show that the map has infinitely many aperiodic orbits.

   (e) By considering the rate of separation of nearby orbits, show that the map has sensitive dependence on initial conditions, and hence displays chaos (since it has aperiodic orbits with sensitive dependence on initial conditions).

   (f) For the case $b = 2$, show that the quantities $y_n = \sin^2 \pi x_n$ satisfy $y_{n+1} = 4y_n(1 - y_n)$ for all $0 \le y_n \le 1$, if x_n are iterates of the shift map. Combined with the result of part (d), this constitutes a proof that the logistic map is chaotic at $r = 4$.

See over →

4. **Extra credit (i.e., optional, but kind of fun):** Write a program in your favorite computer language (C, Perl, Matlab, Mathematica, etc.) to create a plot of the famous "figtree" orbit diagram of the logistic map—the plot of the attractor as a function of r. Attach a copy of your program and the plot to your homework for extra brownie points.