Complex Systems 899: Homework 7

1. **Exponent of the wealth distribution:** Consider the combination-of-exponentials mechanism for generating power laws of Reed and Hughes, as described in the handout.

   (a) The typical rate of return on stock market investments in the US, averaged over the last 70 years, is about 11% per year. Over the same period life expectancy was about 70 years. From these figures, make an estimate of the exponent α of the distribution of people's wealth at the time they die, assuming the probability per year of a person dying is a constant over their lifetime.

   (b) This last assumption is a pretty bad one—in fact people are much more likely to die later in life than earlier. In which direction would you expect the exponent to change if you accounted for this bias and why?

2. **Random walks:**

   (a) Write down an expression for the probability that a random walk in one dimension on a regular grid, starting at the origin, is back at the origin a time $t = 2n$ later, with n integer. Make an approximation to this expression using Stirling's formula $\ln n! \simeq n \ln n - n + \frac{1}{2}\ln(2\pi n)$. Hence calculate an approximate figure for the expected number of times we will revisit the origin in total time T. (You can't do the sum in closed form, but you can approximate it by an integral with reasonable results.)

   (b) Now suppose we do a random walk in two dimensions, on a square grid. On each step we *both* take a step east/west and a step north/south. This two-dimensional random walk is just the "product" of two independent one-dimensional walks along the two axes of the grid. Hence calculate the probability that, if we start at the origin, we are back there again at time $t = 2n$. From this make a statement about how the total number of times we revisit the origin scales as the total time $T \to \infty$.

   (c) Do the same in three dimensions.

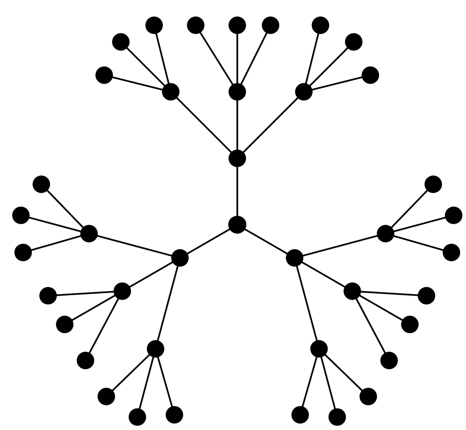
   (d) In simple words, what can we then say about visits to the origin for random walks in one, two, and three dimensions?

3. **Percolation and forest fires:** Let s be the size of a "cluster" of connected squares for site percolation on a square lattice. Precisely at the critical point, the fraction of clusters with size s is $P(s) \sim s^{-\alpha}$ with $\alpha \simeq 2.1$.

   (a) Consider a simple model of a forest in which trees grow on a square grid at random. A fire starts at a random point in the forest, but can only spread between trees on adjacent squares. Thus the fire burns only the cluster in which it starts. When the forest is exactly at the critical point, what is the shape of the tail of probability distribution of the size of fires? What does this say about the mean fire size?

(b) We are tree farmers. Fires are bad, so this random forest is not a good thing. We come up with a better plan. We grow trees on a large grid and cut firebreaks one grid-square wide to prevent fires from spreading. Suppose we cut the breaks themselves in a grid-like fashion—north/south and east/west, with a spacing of d grid squares between parallel firebreaks—and all other grid squares are used for growing trees. Fires start at random positions at a rate of r per grid square per year and trees take time $1/a$ years to grow to maturity and be harvested. Assuming $a > 4r$, what spacing d should we choose for the firebreaks in order to optimize our profits?

4. **Percolation on the Cayley tree:** The Cayley tree or Bethe lattice is a tree-like lattice that looks like this:



There is a fixed "branching ratio" b at each point, where the tree splits in b new directions. In the figure above $b = 3$.

Consider bond percolation with probability p on this lattice, in the limit where the branching repeats arbitrarily and the lattice has infinite size. At some value p_c of the bond probability p a giant cluster will form on the lattice. Imagine following one of the bonds on the lattice "outwards" from the center and let S be the probability that the site we reach by doing this belongs to the giant cluster. If that site *doesn't* belong to the infinite cluster that means that for each of the b sites one step further out from it either (a) the connecting bond is not occupied or (b) the bond is occupied but the site it connects to is itself not a member of the giant cluster.

(a) Express this statement mathematically as an equation for S involving p and the branching ratio b.

(b) Solve this equation for S in the case $b = 3$ and hence show that the percolation transition occurs at $p_c = \frac{1}{3}$.