Complex Systems 899: Homework 8

1. **The Yule distribution:** We have seen that a suitable form for a power-law distribution for integer variables is the so-called Yule distribution:

$$p(k) = C\,\mathrm{B}(k,\alpha) = C\,\frac{\Gamma(k)\Gamma(\alpha)}{\Gamma(k+\alpha)}, \tag{1}$$

where C is a normalizing constant and $\Gamma(x)$ is the standard Γ-function, which has the property $\Gamma(x+1) = x\Gamma(x)$.

   (a) Consider the integral

$$I(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u.$$

   Show, using integration by parts, that for $a > 1$ and $b > 0$,

$$I(a,b) = \frac{a-1}{b} I(a-1, b+1).$$

   Hence, by repeated use of this formula, show that for k integer

$$\mathrm{B}(k,\alpha) = \int_0^1 u^{k-1}(1-u)^{\alpha-1}\,\mathrm{d}u. \tag{2}$$

   (b) Now that we have an integral form for the beta-function, we can show some useful properties of the Yule distribution. First, from the normalization condition $\sum_{k=0}^{\infty} p(k) = 1$, show that the normalizing constant C in Eq. (1) takes the value $C = \alpha - 1$.

   (c) Show that

$$\langle k \rangle = \sum_{k=0}^{\infty} k p(k) = \frac{\alpha-1}{\alpha-2}.$$

   (d) Make the change of variables $y = k(1-u)$ and hence show that in the limit of large k, $p(k)$ does indeed have a power-law tail thus:

$$p(k) \simeq (\alpha-1)\Gamma(\alpha)k^{-\alpha}.$$

2. **Critical exponents:** In class we saw how a simple real-space renormalization transformation $p \to p'$ can give an approximate solution for the position of the phase transition in site percolation on the square lattice. We found that $p_c \simeq \frac{1}{2}(\sqrt{5}-1)$. You can also use the renormalization method to calculate other parameters of the phase transition. For instance, we know that the average size $\langle s \rangle$ of the cluster to which a randomly chosen site belongs diverges as $p \to p_c$. Let us assume it does so as

$$\langle s \rangle \sim (p_c - p)^{-\beta},$$

where β is a so-called "critical exponent."

Our renormalization transformation coarse-grains the system so that $\langle s\rangle \to b\langle s\rangle$, with $b = \frac{1}{4}$ in this case, and $p \to p'$, where we calculated p' in class. Show that, within the approximation made by this renormalization transformation,

$$\beta = \frac{\ln 4}{\ln 2\left(3 - \sqrt{5}\right)}.$$

3. **The optimal location problem:** We saw in class that the problem of cutting a forest into patches to minimize the cost of fires gives rise to a power law in the fire sizes. Here is another example of a similar process.

   Consider the problem of locating n facilities, such as airports, hospitals, or stores, within a country so that the average distance a member of population has to go to reach the nearest one is minimized. Each facility will serve a certain patch or region—the set of points that are nearer to it than to any other facility. (Technically, this is called a *Voronoi cell*.) Let $s(\mathbf{r})$ be the size of the patch to which the point $\mathbf{r}$ belongs, just as in the forest fire example. You can consider the Earth to be two-dimensional and flat.

   (a) Write down a rough expression for the typical distance a person living in a patch of size s will have to go to reach their nearest facility. (This is similar to the expression we wrote for the perimeter of a patch in the forest fire model: this is the "cost" for the patch. It will contain an unknown geometric factor that depends on the shape of the patch. You can assume that this factor is constant, which is an approximation.)

   (b) Note that $s(\mathbf{r})$ is constant in a patch. Show that the integral $\int [s(\mathbf{r})]^{-1}\, \mathrm{d}^2 r$ over any one patch just equals 1. Hence find an expression for this integral over all patches.

   (c) You are given the population density $\rho(\mathbf{r})$ of the country as a function of position. Using the method of Lagrange multipliers, or otherwise, find the patch size $s(\mathbf{r})$ as a function of $\rho(\mathbf{r})$ that minimizes the average distance from a member of the population to their nearest facility, within the approximation described here and with the total number n of facilities held fixed. You should assume that $s(\mathbf{r})$ can take any functional form (just as we did with the case of the forest fires) even though strictly it would be piecewise constant. Hence show that the optimal density $\mathscr{D}(\mathbf{r})$ of facilities approximately satisfies the power-law relation $\mathscr{D}(\mathbf{r}) \propto [\rho(\mathbf{r})]^{2/3}$.

   This problem is an example of *allometric scaling*, which has become a particularly popular topic of study in ecology and related fields in recent years. Allometric scaling laws, such as the one derived here, are seen in a huge variety of phenomena in the natural world as well as the human one.