Complex Systems 511: Homework 3

1. **An example of a 2D nonlinear system:** Consider the system defined by the equations $\dot{x} = y - 2x$, $\dot{y} = \mu + x^2 - y$, where μ is a parameter.

   (a) Solve for the positions of the fixed point(s).

   (b) Classify the bifurcation(s) that occur as μ varies.

   (c) Give a sketch of the phase portrait.

2. **Hopf bifurcation in a predator-prey model:** Let x, y be proportional to the populations of a predator and a prey in an ecosystem. The equations

$$\dot{x} = x^2(1 - x) - xy, \qquad \dot{y} = xy - ay,$$

   have been proposed as describing the dynamics, with $a > 0$.

   (a) Give a brief interpretation of the equations in population dynamics terms. You can start off by saying which of x, y is the predator population and which is the prey.

   (b) Show that the fixed points are at $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$ and classify the behavior at those points to linear order for values of a close to 1.

   (c) Demonstrate that the predators always go extinct if $a > 1$.

   (d) Show that a Hopf bifurcation occurs at $a_c = \frac{1}{2}$.

   (e) Estimate the frequency of the population oscillations near the bifurcation in terms of the parameter a.

3. **Hopf bifurcation in the Lorenz equations:** There are three fixed points in the Lorenz equations for $r > 1$: the origin and the two points C^+ and C^-.

   (a) Find the Jacobian at the fixed points C^+ and C^- and show that its characteristic equation, which is now cubic because this is a 3D system, is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

   (b) A Hopf bifurcation occurs when eigenvalues cross the imaginary line—i.e., when they are pure imaginary. Look for purely imaginary solutions $\lambda = i\omega$ to the equation above (with ω real) and hence show that there is a Hopf bifurcation when

$$r = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

   (c) You'll need to assume $\sigma > b - 1$. Why?

   (d) Find the third eigenvalue.