Complex Systems 511: Homework 7

1. **Random walks:**

   (a) Write down an expression for the probability that a random walk in one dimension on a regular grid, starting at the origin, is back at the origin a time $t = 2n$ later, with n integer. Make an approximation to this expression using Stirling's formula $\ln n! \simeq n \ln n - n + \frac{1}{2}\ln(2\pi n)$. Hence calculate an approximate figure for the expected number of times we will revisit the origin in total time T. (You can't do the sum in closed form, but you can approximate it by an integral with reasonable results.)

   (b) Now suppose we do a random walk in two dimensions on a square grid. On each step we take *both* a step east/west and a step north/south. This two-dimensional random walk is just the "product" of two independent one-dimensional walks along the two axes of the grid. Hence calculate the probability that, if we start at the origin, we are back there again at time $t = 2n$. From this make a statement about how the total number of times we revisit the origin scales as the total time $T \to \infty$.

   (c) Do the same in three dimensions.

   (d) In simple words, what can we then say about visits to the origin for random walks in one, two, and three dimensions?

2. **The Yule distribution:** In the solution of the Yule process we derived the limiting distribution we called the Yule distribution:

$$p(k) = C\,\mathrm{B}(k, \alpha) = C\,\frac{\Gamma(k)\Gamma(\alpha)}{\Gamma(k + \alpha)}, \tag{1}$$

   where C is a normalizing constant and $\Gamma(x)$ is the standard gamma function:

$$\Gamma(x) = \int_0^\infty s^{x-1} \mathrm{e}^{-s}\, \mathrm{d}s.$$

   (a) Employing this integral form for the gamma function write the beta function $\mathrm{B}(a, b)$ as a double integral over s and t. (You can leave the gamma in the denominator as it is for the moment.) Change variables to $u = s/(s + t)$, $v = s + t$ (taking care to get the Jacobian and limits right) and hence show that

$$\mathrm{B}(a, b) = \int_0^1 u^{a-1}(1 - u)^{b-1}\, \mathrm{d}u.$$

   (b) Now that we have an integral form for the beta-function, we can show some useful properties of the Yule distribution. First, from the normalization condition $\sum_{k=1}^{\infty} p(k) = 1$, show that the normalizing constant C in Eq. (1) takes the value $C = \alpha - 1$.

(c) Show that

$$\langle k \rangle = \sum_{k=1}^{\infty} k p(k) = \frac{\alpha - 1}{\alpha - 2}.$$

(d) Make the change of variables $y = k(1-u)$ and hence show that in the limit of large k, $p(k)$ does indeed have a power-law tail thus:

$$p(k) \simeq (\alpha - 1)\Gamma(\alpha) k^{-\alpha}.$$

3. **Critical exponents:** In class we saw how a simple real-space renormalization transformation $p \to p'$ can give an approximate solution for the position of the phase transition in site percolation on the square lattice. We found that $p_c \simeq \frac{1}{2}(\sqrt{5} - 1)$. You can also use the renormalization method to calculate other parameters of the phase transition. For instance, we know that the average size $\langle s \rangle$ of the cluster to which a randomly chosen site belongs diverges as $p \to p_c$. Let us assume it does so as

$$\langle s \rangle \sim (p_c - p)^{-\beta},$$

where β is a so-called "critical exponent."

Our renormalization transformation coarse-grains the system so that $\langle s \rangle \to b\langle s \rangle$, with $b = \frac{1}{4}$ in this case, and $p \to p'$, where we calculated p' in class. Show that, within the approximation made by this renormalization transformation,

$$\beta = \frac{\ln 4}{\ln 2\left(3 - \sqrt{5}\right)}.$$