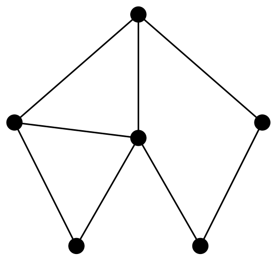
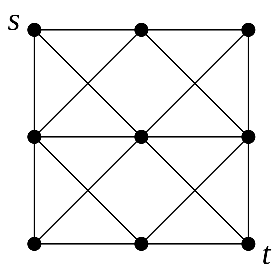
Complex Systems 535/Physics 508: Final Exam sample questions

1. **Four small graphs:**

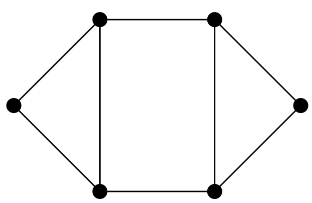
   (a) [**3 points**] Does this graph contain an Eulerian path? (If so, draw the path.) Does it contain a Hamiltonian path? (You don't need to draw the Hamiltonian path.)

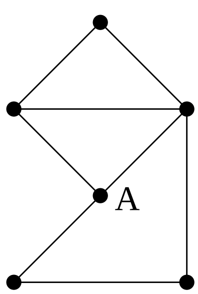
   (b) [**3 points**] What is the edge connectivity of vertices *s* and *t* in this graph? Draw on the figure to illustrate your answer. What is the vertex connectivity of *s* and *t*?

   (c) [**2 points**] Circle the 2-cliques in this graph:

   (d) [**2 points**] What is the local clustering coefficient of vertex A in this graph?

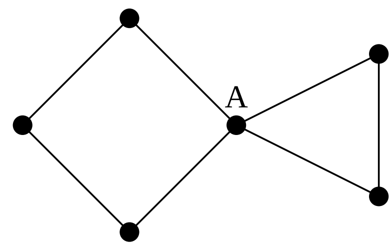
   

2. **Centrality and algorithms:**

   (a) [**3 points**] Define the degree centrality, closeness centrality, and betweenness centrality of a vertex in an undirected network.

   (b) [**3 points**] For an undirected graph of *m* edges and *n* vertices in **adjacency list** form, give the leading-order computational complexity, in terms of *m* and *n*, of the calculation of each of these centralities for a single vertex.

(c) [**4 points**] Calculate each of these three centralities for the vertex A in the center of this network:



3. **Algebraic connectivity:**

   (a) [**1 point**] Write down the definition of the graph Laplacian $\mathbf{L}$ of an undirected graph.

   (b) [**1 point**] Write down the definition of the edge incidence matrix $\mathbf{B}$ of an undirected graph (the matrix for which $\mathbf{L} = \mathbf{B}^T\mathbf{B}$).

   (c) [**3 points**] Given that $\mathbf{L} = \mathbf{B}^T\mathbf{B}$, show that all eigenvalues of the Laplacian are non-negative.

   (d) [**2 points**] Show that the vector $(1,1,1,\ldots)$ is the eigenvector of the Laplacian with the lowest eigenvalue.

   (e) [**3 points**] Hence argue that the algebraic connectivity of an undirected network is zero if the network has more than one component.

4. **Giant component in a random graph:** Consider a random graph with specified degree distribution in the limit of large size. The degree distribution is given by $p_k = Ca^k$ for $k \geq 0$, where $0 < a < 1$ and C is a normalizing constant.

   (a) [**2 points**] Find an expression for C in terms of a.

   (b) [**4 points**] Find a closed-form expression for the probability generating function for the degree distribution.

   (c) [**4 points**] Hence or otherwise find a condition on a—an inequality—that is satisfied if and only if there is a giant component in the network.

5. **Clustering coefficient on a random graph:** A graph of n vertices is constructed as follows. Each vertex i is assigned a "fugacity" λ_i in the range $0 \leq \lambda_i \leq 1$, and each vertex pair (i, j) has an edge connecting it with probability proportional to $\lambda_i\lambda_j$.

   Calculate the following, in terms of the mean $\langle\lambda\rangle$ and mean square $\langle\lambda^2\rangle$ of the fugacities, and the number of vertices n:

   (a) [**2 points**] The expected degree of vertex i.

   (b) [**3 points**] The expected number of triangles in the network for large n.

   (c) [**3 points**] The expected number of connected triples for large n, i.e., unordered pairs of vertices both connected to another common vertex. (If a particular pair is connected to two other common vertices, count that as two connected triples.)

   (d) [**2 points**] The clustering coefficient for this network in the limit of large n.