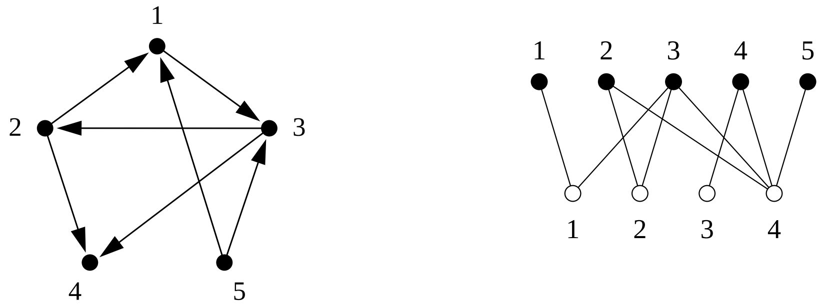
Complex Systems 535/Physics 508: Homework 1

1. Consider the following two networks:



(a) (b)

Network (a) is a directed network. Network (b) is undirected but bipartite. Write down:

(i) the adjacency matrix of network (a);

(ii) the cocitation matrix of network (a);

(iii) the incidence matrix of network (b);

(iv) the projection matrix for the projection of network (b) onto its black vertices.

2. Consider an acyclic directed network of n vertices, labeled $i = 1 \ldots n$, and suppose the labels are assigned in the manner depicted in Fig. 6.3 of the course-pack, such that all edges run from vertices with higher labels to vertices with lower.

(i) Find an expression for the total number of edges ingoing to vertices $1 \ldots r$ and another for the total number of edges outgoing from vertices $1 \ldots r$, in terms of the in- and out-degrees k_i^{in} and k_i^{out} of the vertices.

(ii) Hence find an expression for the total number of edges running to vertices $1 \ldots r$ from vertices $r+1 \ldots n$.

(iii) Show that in any acyclic network the in- and out-degrees must satisfy

$$k_{r+1}^{\text{out}} \leq \sum_{i=1}^{r} (k_i^{\text{in}} - k_i^{\text{out}}),$$

for all $r = 1 \ldots n-1$.

3. Consider a bipartite network, with its two types of vertices, and suppose there are n_1 vertices of type 1 and n_2 vertices of type 2. Show that the mean degrees c_1 and c_2 of the two types are related by

$$c_2 = \frac{n_1}{n_2} c_1.$$

4. Consider the set of all paths from vertex s to vertex t on an undirected graph with adjacency matrix $\mathbf{A}$. Let us give each path a weight equal to α^r, where r is the length of the path and α is a non-negative real constant.

   (i) Show that the sum of the weights of all the paths from s to t is given by Z_{st} which is the st element of the matrix $\mathbf{Z} = (\mathbf{I} - \alpha\mathbf{A})^{-1}$, where $\mathbf{I}$ is the identity matrix.

   (ii) What condition must α satisfy for the sum to converge?

   (iii) Hence, or otherwise, show that the length ℓ_{st} of a geodesic path from s to t, if there is one, is

$$\ell_{st} = \lim_{\alpha \to 0} \frac{\partial \log Z_{st}}{\partial \log \alpha}.$$

5. One possible definition of the trophic level x_i of a species in a (directed) food web is given in Section 5.3.1 of the course-pack, as the mean of the trophic levels of the species' prey, plus one.

   (i) Show that x_i, when defined in this way, is the ith element of the vector

$$\mathbf{x} = (\mathbf{D} - \mathbf{A})^{-1} \mathbf{D} \cdot \mathbf{1},$$

   where $\mathbf{D}$ is the diagonal matrix of in-degrees, $\mathbf{A}$ is the (asymmetric) adjacency matrix, and $\mathbf{1} = (1, 1, 1, \ldots)$.

   (ii) This expression does not work for autotrophs—species with no prey—because the corresponding vector element diverges. Such species are usually given trophic level of 1. Suggest a modification of the calculation that will correctly assign trophic levels to these species, and hence to all species.