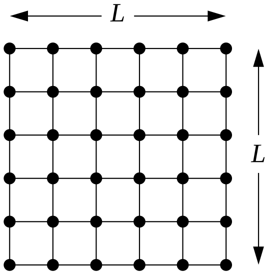
Complex Systems 535/Physics 508: Homework 3

1. One can calculate the diameter of certain types of networks exactly:

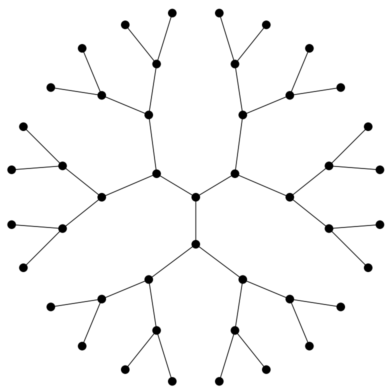
   (i) What is the diameter of a clique?

   (ii) What is the diameter of a square portion of square lattice, with L edges (or equivalently $L+1$ vertices) along each side, like this:

   What is the diameter of the corresponding hypercubic lattice in d dimensions with L edges along each side? Hence what is the diameter of such a lattice as a function of the number n of vertices?

   (iii) A Cayley tree is a symmetric regular tree in which each vertex is connected to the same number k of others, until we get out to the leaves, like this:

   (We have $k = 3$ in this picture.)

   Show that the number of vertices reachable in d steps from the central vertex is $k(k-1)^{d-1}$ for $d \geq 1$. Hence find an expression for the diameter of the network in terms of k and the number of vertices n.

   (iv) Which of the networks in parts (i), (ii), and (iii) displays the small-world effect, defined as having a diameter that increases as $\log n$ or slower?

2. Suppose a network has a degree distribution that follows the exponential form $p_k = Ce^{-\lambda k}$, where C and λ are constants.

   (i) Find C as a function of λ.

   (ii) Calculate the fraction P of vertices that have degrees greater than or equal to k.

   (iii) Calculate the fraction W of ends of edges that are attached to vertices of degree greater than or equal to k.

(iv) Hence show that for this degree distribution the Lorenz curve—the equivalent of Eq. (8.23) in the book—is given by

$$W = P + \frac{1 - e^{\lambda}}{\lambda} P \ln P.$$

(v) What is the equivalent of the "80–20" rule for such a network with $\lambda = 1$? That is, what fraction of the "richest" nodes in the network have 80% of the "wealth"?

(vi) Show that the value of W is greater than one for some values of P in the range $0 \leq P \leq 1$. What is the meaning of these "unphysical" values?

3. A particular network is believed to have a degree distribution that follows a power law. A random sample of vertices is taken and their degrees measured. The degrees of the first twenty vertices with degrees 10 or greater are:

16	17	10	26	13
14	28	45	10	12
12	10	136	16	25
36	12	14	22	10

Estimate the exponent α of the power law and the error on that estimate.

4. Consider the following simple and rather unrealistic model of a network: each of n vertices belongs to one of g groups. The mth group has n_m vertices and each vertex in that group is connected to others in the group with independent probability $p_m = A(n_m - 1)^{-\beta}$, where A and β are constants, but not to any vertices in other groups. Thus this network takes the form of a set of disjoint groups or communities.

(i) Calculate the expected degree $\langle k \rangle$ of a vertex in group m.

(ii) Calculate the expected value $\overline{C}_m$ of the local clustering coefficient for vertices in group m.

(iii) Hence show that $\overline{C}_m \propto \langle k \rangle^{-\beta/(1-\beta)}$. What value would β have to have for the expected value of the local clustering to fall off as $\langle k \rangle^{-0.75}$, as has been conjectured by some researchers?