Complex Systems 535/Physics 508: Homework 6

1. Consider the binomial probability distribution $p_k = \binom{n}{k} p^k (1-p)^{n-k}$.

   (i) Show that the distribution has probability generating function $g(z) = (pz + 1 - p)^n$.

   (ii) Find the first and second moments of the distribution from Eq. (13.25) and hence show that the variance of the distribution is $\sigma^2 = np(1-p)$.

   (iii) Show that the sum of two numbers drawn independently from the same binomial distribution is distributed according to $\binom{2n}{k} p^k (1-p)^{2n-k}$.

2. Consider the configuration model with exponential degree distribution $p_k = (1 - e^{-\lambda})e^{-\lambda k}$ with $\lambda > 0$, so that the generating functions $g_0(z)$ and $g_1(z)$ are given by Eq. (13.130).

   (i) Show that the probability u in Eq. (13.91) satisfies the cubic equation $u^3 - 2e^{\lambda}u^2 + e^{2\lambda}u - (e^{\lambda} - 1)^2 = 0$.

   (ii) Noting that $u = 1$ is always a trivial solution of Eq. (13.91), show that the nontrivial solution corresponding to the existence of a giant component satisfies the quadratic equation $u^2 - (2e^{\lambda} - 1)u + (e^{\lambda} - 1)^2 = 0$, and hence that the size of the giant component, if there is one, is

   $$S = \tfrac{3}{2} - \sqrt{e^{\lambda} - \tfrac{3}{4}}.$$

   Roughly sketch the form of S as a function of λ.

   (iii) Show that the giant component exists only if $\lambda < \ln 3$.

3. Consider the example model discussed in Section 13.8.1, a configuration model with vertices of degree three and less only and generating functions given by Eqs. (13.94) and (13.95).

   (i) In the regime in which there is no giant component, show that the average size of the component to which a randomly chosen vertex belongs is

   $$\langle s \rangle = 1 + \frac{(p_1 + 2p_2 + 3p_3)^2}{p_1 - 3p_3}.$$

   (ii) In the same regime find the probability that such a vertex belongs to components of size 1, 2, and 3.

4. The Internet is found to have a power-law degree distribution $p_k \sim k^{-\alpha}$, with $\alpha \simeq 2.5$ and $k \geq 1$.

   (i) Make a mathematical model of the Internet using the configuration model with this degree distribution. Write down the fundamental generating functions g_0 and g_1.

   (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they can actually send data over the network to each other).