## Complex Systems 535/Physics 508: Homework 3

1. One can calculate the diameter of certain types of networks exactly:
(i) What is the diameter of a clique?
(ii) What is the diameter of a square portion of square lattice, with $L$ edges (or equivalently $L+1$ vertices) along each side, like this:


What is the diameter of the corresponding hypercubic lattice in $d$ dimensions with $L$ edges along each side? Hence what is the diameter of such a lattice as a function of the number $n$ of vertices?
(iii) A Cayley tree is a symmetric regular tree in which each vertex is connected to the same number $k$ of others, until we get out to the leaves, like this:

(We have $k=3$ in this picture.)
Show that the number of vertices reachable in $d$ steps from the central vertex is $k(k-$ $1)^{d-1}$ for $d \geq 1$. Hence find an expression for the diameter of the network in terms of $k$ and the number of vertices $n$.
(iv) Which of the networks in parts (i), (ii), and (iii) displays the small-world effect, defined as having a diameter that increases as $\log n$ or slower?
2. Suppose a network has a degree distribution that follows the exponential form $p_{k}=$ $C \mathrm{e}^{-\lambda k}$, where $C$ and $\lambda$ are constants.
(i) Find $C$ as a function of $\lambda$.
(ii) Calculate the fraction $P$ of vertices that have degrees greater than or equal to $k$.
(iii) Calculate the fraction $W$ of ends of edges that are attached to vertices of degree greater than or equal to $k$.
(iv) Hence show that for this degree distribution the Lorenz curve-the equivalent of Eq. (8.23) in the book-is given by

$$
W=P+\frac{1-\mathrm{e}^{\lambda}}{\lambda} P \ln P
$$

(v) What is the equivalent of the " $80-20$ " rule for such a network with $\lambda=1$ ? That is, what fraction of the "richest" nodes in the network have $80 \%$ of the "wealth"?
(vi) Show that the value of $W$ is greater than one for some values of $P$ in the range $0 \leq P \leq 1$. What is the meaning of these "unphysical" values?
3. A particular network is believed to have a degree distribution that follows a power law. A random sample of vertices is taken and their degrees measured. The degrees of the first twenty vertices with degrees 10 or greater are:

| 16 | 17 | 10 | 26 | 13 |
| ---: | ---: | ---: | ---: | ---: |
| 14 | 28 | 45 | 10 | 12 |
| 12 | 10 | 136 | 16 | 25 |
| 36 | 12 | 14 | 22 | 10 |

Estimate the exponent $\alpha$ of the power law and the error on that estimate.
4. Consider the following simple and rather unrealistic model of a network: each of $n$ vertices belongs to one of $g$ groups. The $m$ th group has $n_{m}$ vertices and each vertex in that group is connected to others in the group with independent probability $p_{m}=$ $A\left(n_{m}-1\right)^{-\beta}$, where $A$ and $\beta$ are constants, but not to any vertices in other groups. Thus this network takes the form of a set of disjoint groups or communities.
(i) Calculate the expected degree $\langle k\rangle$ of a vertex in group $m$.
(ii) Calculate the expected value $\bar{C}_{m}$ of the local clustering coefficient for vertices in group $m$.
(iii) Hence show that $\bar{C}_{m} \propto\langle k\rangle^{-\beta /(1-\beta)}$. What value would $\beta$ have to have for the expected value of the local clustering to fall off as $\langle k\rangle^{-0.75}$, as has been conjectured by some researchers?

