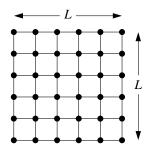
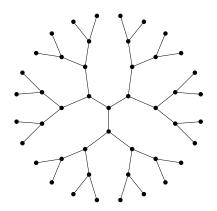
## **Complex Systems 535/Physics 508: Homework 3**

- 1. One can calculate the diameter of certain types of networks exactly:
  - (i) What is the diameter of a clique?
  - (ii) What is the diameter of a square portion of square lattice, with L edges (or equivalently L + 1 vertices) along each side, like this:



What is the diameter of the corresponding hypercubic lattice in d dimensions with L edges along each side? Hence what is the diameter of such a lattice as a function of the number n of vertices?

(iii) A Cayley tree is a symmetric regular tree in which each vertex is connected to the same number *k* of others, until we get out to the leaves, like this:



(We have k = 3 in this picture.)

Show that the number of vertices reachable in *d* steps from the central vertex is  $k(k-1)^{d-1}$  for  $d \ge 1$ . Hence find an expression for the diameter of the network in terms of *k* and the number of vertices *n*.

- (iv) Which of the networks in parts (i), (ii), and (iii) displays the small-world effect, defined as having a diameter that increases as log *n* or slower?
- 2. Suppose a network has a degree distribution that follows the exponential form  $p_k = Ce^{-\lambda k}$ , where *C* and  $\lambda$  are constants.
  - (i) Find *C* as a function of  $\lambda$ .
  - (ii) Calculate the fraction *P* of vertices that have degrees greater than or equal to *k*.
  - (iii) Calculate the fraction *W* of ends of edges that are attached to vertices of degree greater than or equal to *k*.

(iv) Hence show that for this degree distribution the Lorenz curve—the equivalent of Eq. (8.23) in the book—is given by

$$W = P + \frac{1 - e^{\lambda}}{\lambda} P \ln P.$$

- (v) What is the equivalent of the "80–20" rule for such a network with  $\lambda = 1$ ? That is, what fraction of the "richest" nodes in the network have 80% of the "wealth"?
- (vi) Show that the value of *W* is greater than one for some values of *P* in the range  $0 \le P \le 1$ . What is the meaning of these "unphysical" values?
- 3. A particular network is believed to have a degree distribution that follows a power law. A random sample of vertices is taken and their degrees measured. The degrees of the first twenty vertices with degrees 10 or greater are:

16	17	10	26	13
14	28	45	10	12
12	10	136	16	25
36	12	14	22	10

Estimate the exponent  $\alpha$  of the power law and the error on that estimate.

- 4. Consider the following simple and rather unrealistic model of a network: each of n vertices belongs to one of g groups. The mth group has  $n_m$  vertices and each vertex in that group is connected to others in the group with independent probability  $p_m = A(n_m 1)^{-\beta}$ , where A and  $\beta$  are constants, but not to any vertices in other groups. Thus this network takes the form of a set of disjoint groups or communities.
  - (i) Calculate the expected degree  $\langle k \rangle$  of a vertex in group *m*.
  - (ii) Calculate the expected value  $\overline{C}_m$  of the local clustering coefficient for vertices in group *m*.
  - (iii) Hence show that  $\overline{C}_m \propto \langle k \rangle^{-\beta/(1-\beta)}$ . What value would  $\beta$  have to have for the expected value of the local clustering to fall off as  $\langle k \rangle^{-0.75}$ , as has been conjectured by some researchers?