Physics 411: Homework 4

1. **Simpson's rule:**

   (a) Write a program to calculate an approximate value for the integral $\int_0^2 (x^4 - 2x + 1)\,\mathrm{d}x$ from Example 5.1, but using Simpson's rule with 10 slices instead of the trapezoidal rule.

   (b) Run the program and compare your result to the known correct value of 4.4. What is the fractional error on your calculation?

   (c) Modify the program to use a hundred slices instead, then a thousand. Note the improvement in the result. How do the results compare with those from Example 5.1 for the trapezoidal rule with the same number of slices?

   ☑ **For full credit** turn in a printout of your program, plus your results and a brief discussion of how they compare with the trapezoidal rule.

2. **Adaptive integration and Romberg integration:**

   (a) Write a program that uses the adaptive trapezoidal rule method of Section 5.3 and Eq. (5.24) to calculate the value of the integral

   $$I = \int_0^1 \sin^2 \sqrt{100x}\,\mathrm{d}x$$

   to an approximate accuracy of $\epsilon = 10^{-6}$ (i.e., correct to six digits after the decimal point). Start with one single integration slice and work up from there to two, four, eight, and so forth. Have your program print out the number of slices, its estimate of the integral, and its estimate of the error on the integral, for each value of the number of slices N, until the target accuracy is reached. (Hint: you should find the result is around $I = 0.45$.)

   (b) Now modify your program to evaluate the same integral using the Romberg integration technique described in Section 5.4, for the same series of values of N. Have your program print out a triangular table of values, as on page 139, of all the Romberg estimates of the integral.

   ☑ **For full credit** turn in a printout of your final program from part (b) and printouts showing both your programs in action (part (a) and part (b)) and clearly showing the output values.

3. **Heat capacity of a solid:** Debye's theory of solids gives the heat capacity of a solid at temperature T to be

$$C_V = 9V\rho k_B \left(\frac{T}{\theta_D}\right)^3 \int_0^{\theta_D/T} \frac{x^4 \mathrm{e}^x}{(\mathrm{e}^x - 1)^2}\,\mathrm{d}x,$$

where V is the volume of the solid, ρ is the number density of atoms, k_B is Boltzmann's constant, and θ_D is the so-called *Debye temperature*, a property of solids that depends on their density and speed of sound.

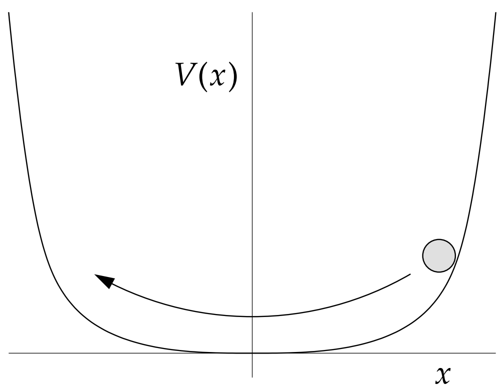
(a) Write a Python function `cv(T)` that calculates C_V for a given value of the temperature, for a sample consisting of 1000 cubic centimeters of solid aluminum, which has a number density of $\rho = 6.022 \times 10^{28}\,\text{m}^{-3}$ and a Debye temperature of $\theta_D = 428\,\text{K}$. Use Gaussian quadrature to evaluate the integral, with $N = 50$ sample points.

(b) Use your function to make a graph of the heat capacity as a function of temperature from $T = 5\,\text{K}$ to $T = 500\,\text{K}$.

☑ **For full credit** turn in a printout of your program and the graph that it produces.

4. **The period of an anharmonic oscillator:** The simple harmonic oscillator crops up in many places. Its behavior can be studied readily using analytic methods and it has the important property that its period of oscillation is a constant, independent of its amplitude, making it useful, for instance, for keeping time in watches and clocks. Frequently in physics, however, we also come across anharmonic oscillators, whose period varies with amplitude and whose behavior cannot usually be calculated analytically.

A general classical oscillator can be thought of as a particle in a concave potential well. When disturbed, the particle will rock back and forth in the well:



The harmonic oscillator corresponds to a quadratic potential $V(x) \propto x^2$. Any other form gives an anharmonic oscillator. (Thus there are many different kinds of anharmonic oscillator, depending on the exact form of the potential.)

One way to calculate the motion of an oscillator is to write down the equation for the conservation of energy in the system. If the particle has mass m and position x, then the total energy is equal to the sum of the kinetic energy $\frac{1}{2}mv^2$ and the potential energy $V(x)$ thus:

$$E = \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x).$$

Since the energy must be constant over time, this equation is effectively a (nonlinear) differential equation linking x and t.

Let us assume that the potential $V(x)$ is symmetric about $x = 0$ and let us set our anharmonic oscillator going with amplitude a. Specifically, at $t = 0$ we release it from rest at position $x = a$ and it swings back toward the origin. Then at $t = 0$ we have $\mathrm{d}x/\mathrm{d}t = 0$ and the equation above reads $E = V(a)$, which gives us the total energy of the particle in terms of the amplitude.

(a) When the particle reaches the origin for the first time, it has gone through one quarter of a period of the oscillator. By rearranging the equation above for $\mathrm{d}x/\mathrm{d}t$ and then integrating with respect to t from 0 to $\frac{1}{4}T$, show that the period T is given by

$$T = \sqrt{8m} \int_0^a \frac{\mathrm{d}x}{\sqrt{V(a) - V(x)}}.$$

(b) Suppose the potential is $V(x) = x^4$ and the mass of the particle is $m = 1$. Write a Python function that calculates the period of the oscillator for given amplitude a using Gaussian quadrature, then use your function to make a graph of the period for amplitudes ranging from $a = 0$ to $a = 2$.

(c) You should find that the oscillator gets faster as the amplitude increases, even though the particle has further to travel for large amplitude. And you should find that the period diverges as the amplitude goes to zero. Briefly, how do you explain these results?

☑ **For full credit** turn in a printout of your program and the graph that it produces, along with your answers for parts (a) and (c).