Complex Systems 535/Physics 508: Homework 4

1. Consider the random graph $G(n, p)$ with average degree c.

   (i) Show that in the limit of large n the expected number of triangles in the network is $\frac{1}{6}c^3$, where c is the mean degree as before. In other words, the number of triangles is constant, neither growing nor vanishing in the limit of large n.

   (ii) Show that the expected number of connected triples in the network (as in Eq. (7.41)) is $\frac{1}{2}nc^2$.

   (iii) Hence calculate the clustering coefficient C, as defined in Eq. (7.41), and confirm that it agrees for large n with the value given in Eq. (12.11).

2. Starting from the generating function $h(z)$ defined in Eq. (12.26), or otherwise, show that

   (i) the mean-square size of the component in a random graph to which a randomly chosen vertex belongs is $1/(1-c)^3$ in the regime where there is no giant component;

   (ii) the mean-square size of a randomly chosen component in the regime with no giant component is $1/[(1-c)(1-\frac{1}{2}c)]$.

   Note that both quantities diverge at the phase transition where the giant component appears.

3. We can make a random graph model of a network with clustering as follows. We take n vertices and go through each distinct trio of three vertices, of which there are $\binom{n}{3}$, and with independent probability p we connect the members of the trio together using three edges to form a triangle, where $p = c/\binom{n-1}{2}$ with c a constant. You can assume that n is very large.

   (i) Show that the mean degree of a vertex in this model network is $2c$.

   (ii) Show that the degree distribution is $p_k = \mathrm{e}^{-c}c^{k/2}/(k/2)!$ if k is even and $p_k = 0$ if k is odd.

   (iii) Show that the clustering coefficient, Eq. (7.41), is $C = 1/(2c+1)$. Note that, by contrast with the ordinary random graph, this clustering coefficient doesn't go to zero as n becomes large.

   (iv) Show that as a fraction of network size, the expected size S of the giant component, if there is one, satisfies $S = 1 - \mathrm{e}^{-cS(2-S)}$.

   (v) What is the value of the clustering coefficient when the giant component fills half of the network?

4. Here's an alternative way of deriving the spectrum of a random $n \times n$ symmetric matrix $\mathbf{B}$ with identically distributed, zero-mean elements having variance $\sigma^2 = c/n$.

   Recall that the spectral density is given by

$$\rho(z) = -\frac{1}{n\pi}\,\mathrm{Im}\big\langle \mathrm{Tr}(z\mathbf{I} - \mathbf{B})^{-1} \big\rangle.$$

Define $\mathbf{X} = z\mathbf{I} - \mathbf{B}$ then divide this matrix into its first $n-1$ rows and columns, and its last row and column thus:

$$\mathbf{X} = \begin{pmatrix} \boxed{\;\mathbf{X}_n\;} & \boxed{\mathbf{a}} \\ \boxed{\;\mathbf{a}^T\;} & x_{nn} \end{pmatrix}, \tag{1}$$

where $\mathbf{X}_n$ indicates the matrix with the nth row and column removed.

(i) Consider the vector $\mathbf{v} = \mathbf{X}^{-1}\mathbf{u}$, where $\mathbf{u} = (0,\ldots,0,1)$. Break $\mathbf{v}$ into its first $n-1$ elements and its last element $\mathbf{v} = (\mathbf{v}_1|v_n)$. Show that $v_n = \left[\mathbf{X}^{-1}\right]_{nn}$ which is the bottom-right element of $\mathbf{X}^{-1}$. Then write out an expression for $\mathbf{u} = \mathbf{X}\mathbf{v}$ to show that

$$\mathbf{X}_n\mathbf{v}_1 + v_n\mathbf{a} = 0, \qquad \mathbf{a}^T\mathbf{v}_1 + x_{nn}v_n = 1,$$

and hence that

$$\left[\mathbf{X}^{-1}\right]_{nn} = v_n = \frac{1}{x_{nn} - \mathbf{a}^T\mathbf{X}_n^{-1}\mathbf{a}}.$$

If we were to remove any other row and column i from the matrix we would get the equivalent result

$$\left[\mathbf{X}^{-1}\right]_{ii} = \frac{1}{x_{ii} - \mathbf{a}^T\mathbf{X}_i^{-1}\mathbf{a}}.$$

This is the *Schur complement formula*.

(ii) Assume that as n becomes large, the distribution of $\left[\mathbf{X}^{-1}\right]_{ii}$ becomes narrowly peaked. (This is true, though we won't prove it here.) Then argue that

$$\left\langle \left[\mathbf{X}^{-1}\right]_{ii} \right\rangle = \frac{1}{z - \left\langle \mathbf{a}^T\mathbf{X}_i^{-1}\mathbf{a} \right\rangle}.$$

(iii) Bearing in mind that the elements of $\mathbf{a}$ are random, zero-mean variables, show that

$$\left\langle \mathbf{a}^T\mathbf{X}_i^{-1}\mathbf{a} \right\rangle = \sigma^2 \operatorname{Tr} \mathbf{X}_i^{-1}.$$

(iv) Assuming $(1/n)\operatorname{Tr}\mathbf{X}^{-1}$ converges to a well-defined limit as $n \to \infty$, put everything together to show that the Cauchy transform $g(z) = (1/n)\left\langle \operatorname{Tr}(z\mathbf{I} - \mathbf{B})^{-1} \right\rangle$ satisfies the quadratic equation $cg^2 - zg + 1 = 0$, and hence that the spectral density is

$$\rho(z) = \frac{\sqrt{4c - z^2}}{2\pi c}.$$

5. **Extra credit:** Write a computer program in the language of your choice that generates a random graph drawn from the model $G(n,m)$ for given values of n and the average degree $c = 2m/n$, then measures the size of its largest component. Use your program to find the size of the largest component in a random graph with $n = 1\,000\,000$ and $c = 2\ln 2 = 1.3863\ldots$ and compare your answer to the analytic prediction for the giant component of $G(n,p)$ with the same parameter values. You should find good agreement, even though the models are not identical. (As we discussed in class, the models become essentially the same for large values of m and n.)