Complex Systems 535/Physics 508: Homework 5

1. Suppose we have m nonnegative integers $k_1 \ldots k_m$ that are drawn independently, each from its own probability distribution: $p_1(k_1)$ is the distribution for k_1, $p_2(k_2)$ is the distribution for k_2, and so forth. Let $g_1(z)$, $g_2(z)$, etc. be the generating functions for these distributions.

   Let π_s be the probability that the sum $\sum_{i=1}^{m} k_m$ has the value s. Show that the generating function for π_s is $h(z) = \prod_i g_i(z)$. Hence, rederive the result we showed in class that the generating function for the sum of m independent draws from the same distribution $p(k)$ is the generating function for a single draw raised to the mth power.

2. Consider the binomial probability distribution $p_k = \binom{n}{k} p^k (1-p)^{n-k}$.

   (i) Show that the distribution has probability generating function $g(z) = (pz + 1 - p)^n$.

   (ii) Find the first and second moments of the distribution from Eq. (13.25) and hence show that the variance of the distribution is $\sigma^2 = np(1-p)$.

   (iii) Show that the sum s of two numbers drawn independently from the same binomial distribution is distributed according to $\binom{2n}{s} p^s (1-p)^{2n-s}$.

3. Consider the configuration model with exponential degree distribution $p_k = (1 - \mathrm{e}^{-\lambda})\mathrm{e}^{-\lambda k}$ with $\lambda > 0$, so that the generating functions $g_0(z)$ and $g_1(z)$ are given by Eq. (13.130).

   (i) Show that the probability u in Eq. (13.91) satisfies the cubic equation $u^3 - 2\mathrm{e}^{\lambda} u^2 + \mathrm{e}^{2\lambda} u - (\mathrm{e}^{\lambda} - 1)^2 = 0$.

   (ii) Noting that $u = 1$ is always a trivial solution of Eq. (13.91), show that the nontrivial solution corresponding to the existence of a giant component satisfies the quadratic equation $u^2 - (2\mathrm{e}^{\lambda} - 1)u + (\mathrm{e}^{\lambda} - 1)^2 = 0$, and hence that the size of the giant component, if there is one, is

   $$S = \tfrac{3}{2} - \sqrt{\mathrm{e}^{\lambda} - \tfrac{3}{4}}.$$

   Roughly sketch the form of S as a function of λ.

   (iii) Show that the giant component exists only if $\lambda < \ln 3$.

4. (i) Let us model the Internet as a configuration model with a perfect power-law degree distribution $p_k \sim k^{-\alpha}$, with $\alpha \simeq 2.5$ and $k \geq 1$. Write down the fundamental generating functions g_0 and g_1.

   (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they belong to the largest component).

5. **Extra credit:** Write a program in the computer language of your choice (or modify your program from last week) to generate a configuration model network in which there are only vertices of degree 1 and 3 and then calculate the size of the largest component.

   (i) Use your program to calculate the largest component for a network with $n = 10\,000$ nodes when $p_1 = 0.6$ and $p_3 = 0.4$ (and $p_k = 0$ for all other values of k).

   (ii) Modify your program to make a graph of the size of the largest component for values of p_1 from 0 to 1 in steps of 0.01. Hence estimate the value of p_1 for the phase transition at which the giant component disappears.