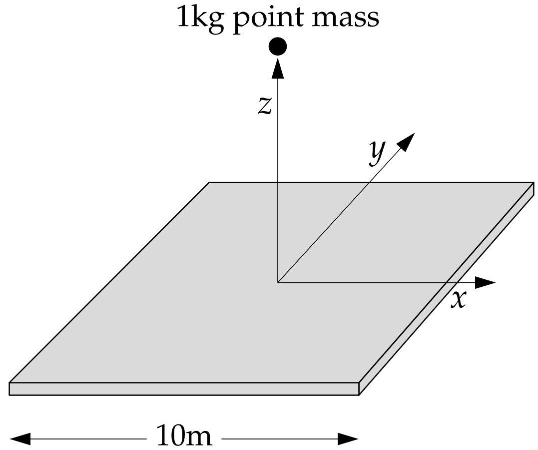
Physics 411: Homework 4

1. **Gravitational pull of a uniform sheet:** A uniform square sheet of metal is floating motionless in space:



The sheet is 10 m on a side and of negligible thickness, and it has a mass of 10 metric tonnes.

(a) Consider the gravitational force due to the plate felt by a point mass of 1 kg a distance z from the center of the square, in the direction perpendicular to the sheet, as shown above. Show that the component of the force along the z-axis is

$$dF_z = G\rho z \iint_{-L/2}^{L/2} \frac{dx\,dy}{(x^2+y^2+z^2)^{3/2}},$$

where $G = 6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is Newton's gravitational constant and ρ is the mass per unit area of the sheet.

(b) Write a program to calculate and plot the force as a function of z from $z = 0$ to $z = 10$ m. For the double integral use (double) Gaussian quadrature, as in Eq. (5.78), with 100 sample points along each axis.

(c) You should see a smooth curve, except at very small values of z, where the force should drop off suddenly to zero. This drop is not a real effect, but an artifact of the way we have done the calculation. Explain briefly where this artifact comes from and suggest a strategy to remove it, or at least to decrease its size.

☑ **For full credit** turn in a printout of your program and the graph that it produces, along with your answer for part (c).

2. **The gamma function:** A commonly occurring function in physics calculations is the gamma function $\Gamma(a)$, which is defined by the integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\,dx.$$

There is no closed-form expression for the gamma function, but one can calculate its value for given a by performing the integral above numerically. You have to be careful how you do it, however, if you wish to get an accurate answer.

(a) Write a program to make a graph of the value of the integrand $x^{a-1}e^{-x}$ as a function of x from $x = 0$ to $x = 5$, with three separate curves for $a = 2$, 3, and 4, all on the same axes. You should find that the integrand starts at zero, rises to a maximum, and then decays again for each curve.

(b) Show analytically that the maximum falls at $x = a - 1$.

(c) Most of the area under the integrand falls near the maximum, so to get an accurate value of the gamma function we need to do a good job of this part of the integral. We can change the integral from 0 to ∞ to one over a finite range from 0 to 1 using the change of variables in Eq. (5.63), but this tends to squash the peak towards the edge of the $[0, 1]$ range and does a poor job of evaluating the integral accurately. We can do a better job by making a different change of variables that puts the peak in the middle of the integration range, around $\frac{1}{2}$. We will use the change of variables given in Eq. (5.65), which we repeat here for convenience:

$$z = \frac{x}{c + x}.$$

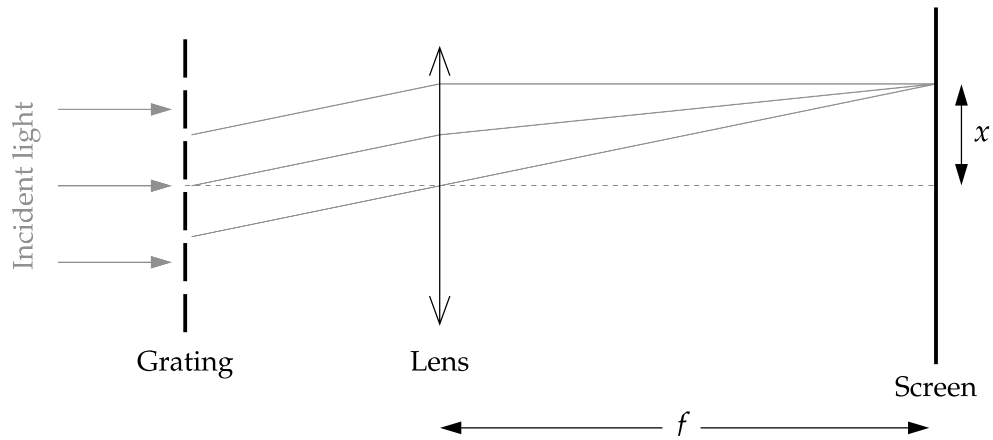
For what value of x does this change of variables give $z = \frac{1}{2}$? Hence what is the appropriate choice of the parameter c that puts the peak of the integrand for the gamma function at $z = \frac{1}{2}$?

(d) Before we can calculate the gamma function, there is another detail we need to attend to. The integrand $x^{a-1}e^{-x}$ can be difficult to evaluate because the factor x^{a-1} can become very large and the factor e^{-x} very small, causing numerical overflow or underflow, or both, for some values of x. Write $x^{a-1} = e^{(a-1)\ln x}$ to derive an alternative expression for the integrand that does not suffer from these problems (or at least not so much). Explain why your new expression is better than the old one.

(e) Now, using the change of variables above and the value of c you have chosen, write a user-defined function `gamma(a)` to calculate the gamma function for arbitrary argument a. Use whatever integration method you feel is appropriate. Test your function by using it to calculate and print the value of $\Gamma(\frac{3}{2})$, which is known to be equal to $\frac{1}{2}\sqrt{\pi} \simeq 0.886$.

☑ **For full credit** turn in a printout of your final program and a printout showing it in action calculating $\Gamma(\frac{3}{2})$, along with your graph from part (a) and your answers to parts (b), (c), and (d).

3. **Diffraction gratings:** Light with wavelength λ is incident on a diffraction grating of total width w, gets diffracted, is focused with a lens of focal length f, and falls on a screen:



Theory tells us that the intensity of the diffraction pattern on the screen, a distance x from the central axis of the system, is given by

$$I(x) = \left| \int_{-w/2}^{w/2} \sqrt{q(u)}\, e^{i2\pi xu/\lambda f}\, du \right|^2,$$

where $q(u)$ is the intensity transmission function of the diffraction grating at a distance u from the central axis.

(a) Consider a grating with transmission function $q(u) = \sin^2 \alpha u$. What is the separation of the "slits" in this grating, expressed in terms of α?

(b) Write a Python function `q(u)` that returns the transmission function $q(u) = \sin^2 \alpha u$ as above at position u for a grating whose slits have separation 20 μm.

(c) Use your function in a program to calculate and graph the intensity of the diffraction pattern produced by such a grating having ten slits in total, if the incident light has wavelength $\lambda = 500$ nm. Assume the lens has a focal length of 1 meter and the screen is 10 cm wide. You can use whatever method you think appropriate for doing the integral. Once you've made your choice you'll also need to decide the number of sample points you'll use. What criteria play into this decision?

Notice that the integrand in the equation for $I(x)$ is complex, so you will have to use complex variables in your program. As mentioned in Section 2.2.5 of the book, there is a version of the `math` package for use with complex variables called `cmath`. In particular you may find the `exp` function from `cmath` useful because it can calculate the exponentials of complex arguments.

(d) Now modify your program to create a visualization of how the diffraction pattern would look on the screen using a density plot. Your plot should look something like this:

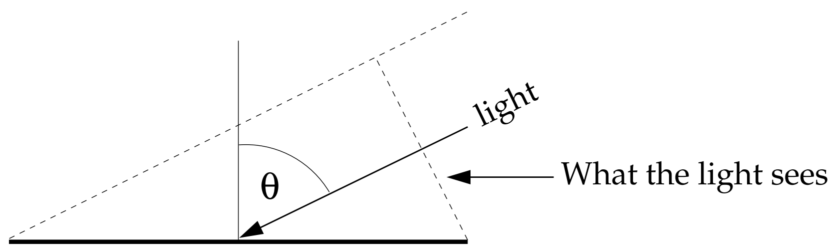
(e) Modify your program further to make pictures of the diffraction patterns produced by gratings with the following profiles:

   i. A transmission profile that obeys $q(u) = \sin^2 \alpha u \sin^2 \beta u$, with α as before and the same total grating width w, and $\beta = \frac{1}{2}\alpha$.

   ii. Two "square" slits, meaning slits with 100% transmission through the slit and 0% transmission everywhere else. Calculate the diffraction pattern for non-identical slits, one 10 μm wide and the other 20 μm wide, with a 60 μm gap between the two.

☑ **For full credit** turn in a printout of your program from part (d), your graph from part (c), and the three density plots from parts (d) and (e).

4. **Image processing and the STM:** When light strikes a surface, the amount falling per unit area depends not only on the intensity of the light, but also on the angle of incidence. If the light makes an angle θ to the normal, it only "sees" $\cos\theta$ of area per unit of actual area on the surface:



So the intensity of illumination is $a\cos\theta$, if a is the raw intensity of the light. This simple physical law is a central element of 3D computer graphics. It allows us to calculate how light falls on three-dimensional objects and hence how they will look when illuminated from various angles.

Suppose, for instance, that we are looking down on the Earth from above and we see mountains. We know the height of the mountains $w(x,y)$ as a function of position in the plane, so the equation for the Earth's surface is simply $z = w(x,y)$, or equivalently $w(x,y) - z = 0$, and the normal vector $\mathbf{v}$ to the surface is given by the gradient of $w(x,y) - z$ thus:

$$\mathbf{v} = \boldsymbol{\nabla}[w(x,y) - z] = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} [w(x,y) - z] = \begin{pmatrix} \partial w/\partial x \\ \partial w/\partial y \\ -1 \end{pmatrix}.$$

Now suppose we have light coming in represented by a vector $\mathbf{a}$ with magnitude equal to the intensity of the light. Then the dot product of the vectors $\mathbf{a}$ and $\mathbf{v}$ is

$$\mathbf{a}\cdot\mathbf{v} = |\mathbf{a}|\,|\mathbf{v}|\cos\theta,$$

where θ is the angle between the vectors. Thus the intensity of illumination of the surface of the mountains is

$$I = |\mathbf{a}| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{a_x(\partial w/\partial x) + a_y(\partial w/\partial y) - a_z}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

Let's take a simple case where the light is shining horizontally with unit intensity, along a line an angle ϕ counter-clockwise from the east-west axis, so that $\mathbf{a} = (\cos\phi, \sin\phi, 0)$. Then our intensity of illumination simplifies to

$$I = \frac{\cos\phi\,(\partial w/\partial x) + \sin\phi\,(\partial w/\partial y)}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

If we can calculate the derivatives of the height $w(x, y)$ and we know ϕ we can calculate the intensity at any point.

(a) In the on-line resources you'll find a file called `altitude.txt`, which contains the altitude $w(x, y)$ in meters above sea level (or depth below sea level) of the surface of the Earth, measured on a grid of points (x, y). Write a program that reads this file and stores the data in an array. Then calculate the derivatives $\partial w/\partial x$ and $\partial w/\partial y$ at each grid point. Explain what method you used to calculate them and why. (Hint: You'll probably have to use more than one method to get every grid point, because awkward things happen at the edges of the grid.) To calculate the derivatives you'll need to know the value of h, the distance in meters between grid points, which is about 30 000 m in this case. (It's actually not precisely constant because we are representing the spherical Earth on a flat map, but $h = 30\,000$ m will give reasonable results.)

(b) Now, using your values for the derivatives, calculate the intensity for each grid point, with $\phi = 45°$, and make a density plot of the resulting values in which the brightness of each dot depends on the corresponding intensity value. If you get it working right, the plot should look like a relief map of the world—you should be able to see the continents and mountain ranges in 3D. (Common problems include a map that is upside-down or sideways, or a relief map that is "inside-out," meaning the high regions look low and *vice versa*. Work with the details of your program until you get a map that looks right to you.)

(c) There is another file in the on-line resources called `stm.txt`, which contains a grid of values from scanning tunneling microscope (STM) measurements of the (111) surface of silicon—the same file that you used in Homework 2. The grid of values in this file represents the height of the silicon surface as a function of position. Modify the program you just wrote to visualize the STM data and hence create a 3D picture of what the silicon surface looks like. The value of h for the derivatives in this case is around $h = 2.5$ (in arbitrary units).

☑ **For full credit** turn in your explanation from part (a), copies of your two density plots, from parts (b) and (c), and your final program from part (c).