

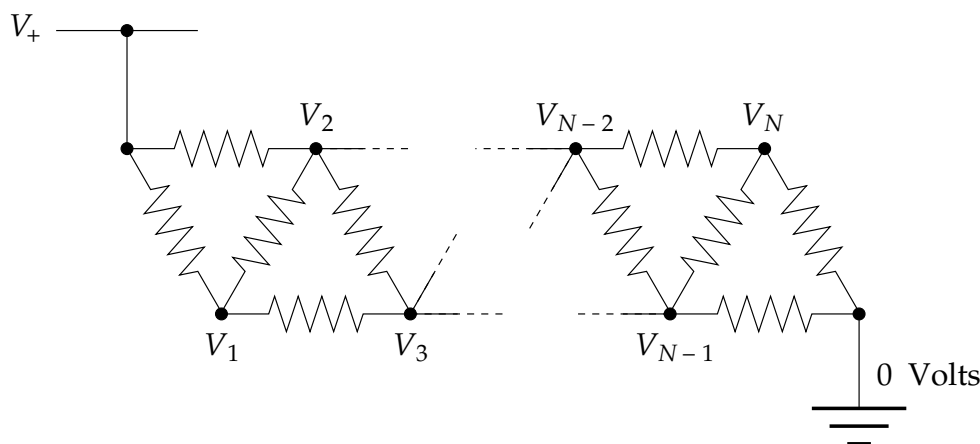
## Physics 411: Homework 5

1. **Pivoting:** Grab yourself a copy of the program `gausselim.py` from the on-line resources.

- Modify the program to incorporate partial pivoting. (You're also welcome to write your own program from scratch if you prefer.) Use your program to solve the problem in Eq. (6.1) and check that it gives the same answer as the original program without pivoting (which is given in Eq. (6.16)).
- Now use your program to solve the equations in Eq. (6.17) and show that it can find the solution to these as well, even though Gaussian elimination without pivoting fails.

✓ **For full credit** turn in a printout of your program, plus your results from part (b).

2. **A circuit of resistors:** Consider a long chain of resistors arranged like this:



All the resistors have the same resistance  $R$ . The power rail at the top is at voltage  $V_+ = 5V$ . The problem is to find the voltages  $V_1 \dots V_N$  at the internal points in the circuit.

- Using Ohm's law and the Kirchhoff current law, which says that the total net current flow out of (or into) any junction in a circuit must be zero, show that the voltages  $V_1 \dots V_N$  satisfy the equations

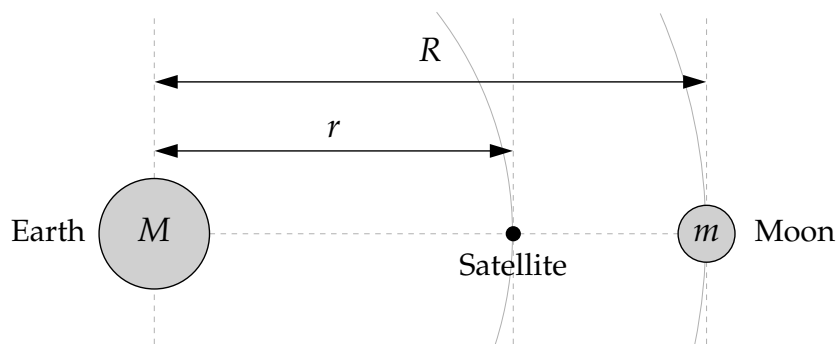
$$\begin{aligned}
 3V_1 - V_2 - V_3 &= V_+, \\
 -V_1 + 4V_2 - V_3 - V_4 &= V_+, \\
 &\vdots \\
 -V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} &= 0, \\
 &\vdots \\
 -V_{N-3} - V_{N-2} + 4V_{N-1} - V_N &= 0, \\
 -V_{N-2} - V_{N-1} + 3V_N &= 0.
 \end{aligned}$$

Express these equations in vector form  $\mathbf{A}\mathbf{v} = \mathbf{w}$  and find the values of the matrix  $\mathbf{A}$  and the vector  $\mathbf{w}$ .

- (b) Write a program to solve for the values of the  $V_i$  when there are  $N = 6$  internal junctions with unknown voltages. Use any method you like to find the solution, including the `solve` function from `numpy` or your own Gaussian elimination code. Make a graph of the values of  $V_i$  along the chain. (Hint: All the values of  $V_i$  should lie between zero and 5V. If they don't, something is wrong.)
- (c) Now repeat your calculation for the case where there are  $N = 10\,000$  internal junctions. This part is not possible using standard tools like the `solve` function. You need to make use of the fact that the matrix  $\mathbf{A}$  is banded and use the banded function that we discussed in class. A copy of this function can be found on the course web page.

✓ **For full credit** turn in your working from part (a), a printout of your final program, and a printout your plot from part (b) for the  $N = 6$  case.

3. **The Lagrange point:** There is a magical point between the Earth and the Moon, called the  $L_1$  Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit. Here's the setup:



- (a) Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, show that the distance  $r$  from the center of the Earth to the  $L_1$  point satisfies

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where  $M$  and  $m$  are the Earth and Moon masses,  $G$  is Newton's gravitational constant, and  $\omega$  is the angular velocity of both the Moon and the satellite.

- (b) The equation above is a fifth-order polynomial equation in  $r$  (also called a quintic equation). Such equations cannot be solved exactly in closed form, but it's straightforward to solve them numerically. Write a program to solve for the distance  $r$  from the Earth to the  $L_1$  point. You can use any of the methods we have studied—binary search, the fixed point method, Newton's method, or the secant method all work

well for this problem. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2},$$

$$M = 5.974 \times 10^{24} \text{ kg},$$

$$m = 7.348 \times 10^{22} \text{ kg},$$

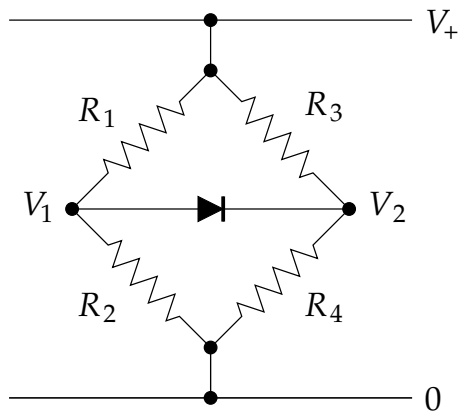
$$R = 3.844 \times 10^8 \text{ m},$$

$$\omega = 2.662 \times 10^{-6} \text{ s}^{-1}.$$

You will also need to choose a suitable starting value for  $r$ , or two starting values if you use the binary search or secant methods.

✓ **For full credit** turn in a copy of your program and your result for the distance  $r$ .

4. **Nonlinear circuits:** Consider the following simple circuit, a variation on the classic Wheatstone bridge:



The resistors obey the normal Ohm law, but the diode obeys the diode equation:

$$I = I_0(e^{V/V_T} - 1),$$

where  $V$  is the voltage across the diode and  $I_0$  and  $V_T$  are constants.

(a) The Kirchhoff current law says that the total net current flowing into or out of every point in a circuit must be zero. Applying the law to voltage  $V_1$  in the circuit above we get

$$\frac{V_1 - V_+}{R_1} + \frac{V_1}{R_2} + I_0[e^{(V_1 - V_2)/V_T} - 1] = 0.$$

Derive the corresponding equation for voltage  $V_2$ .

(b) Solve the two nonlinear equations for the voltages  $V_1$  and  $V_2$  with the conditions

$$\begin{aligned} V_+ &= 5 \text{ V}, \\ R_1 &= 1 \text{ k}\Omega, & R_2 &= 4 \text{ k}\Omega, & R_3 &= 3 \text{ k}\Omega, & R_4 &= 2 \text{ k}\Omega, \\ I_0 &= 3 \text{ nA}, & V_T &= 0.05 \text{ V}. \end{aligned}$$

You can use either the relaxation method or Newton's method to solve the equations. If you use Newton's method you can solve Eq. (6.80) for  $\Delta \mathbf{x}$  using the function `solve()` from `numpy.linalg` if you want to, but in this case the matrix is only a  $2 \times 2$  matrix, so it's easy to calculate the inverse directly too.

(c) The electronic engineer's rule of thumb for diodes is that the voltage across a (forward biased) diode is always about 0.6 volts. Confirm that your results agree with this rule.

✓ **For full credit** turn in your derivation from part (a) and your program from part (b) along with a printout showing the answers it gives.

5. **The temperature of a light bulb:** An incandescent light bulb is a simple device—it contains a resistive filament, usually made of tungsten, heated by the flow of electricity until it becomes hot enough to radiate thermally. Essentially all of the power consumed by such a bulb is radiated as electromagnetic energy, but some of the radiation is not in the visible wavelengths, which means it is useless for lighting purposes.

Let us define the efficiency of a light bulb to be the fraction of the radiated energy that falls in the visible band. It's a good approximation to assume that the radiation obeys the Planck radiation law, meaning that the power radiated per unit wavelength  $\lambda$  obeys

$$I(\lambda) = 2\pi Ahc^2 \frac{\lambda^{-5}}{e^{hc/\lambda k_B T} - 1},$$

where  $A$  is the surface area of the filament,  $T$  is the temperature,  $h$  is Planck's constant,  $c$  is the speed of light, and  $k_B$  is Boltzmann's constant. The visible wavelengths run from  $\lambda_1 = 390 \text{ nm}$  to  $\lambda_2 = 750 \text{ nm}$ , so the total energy radiated in the visible window is  $\int_{\lambda_1}^{\lambda_2} I(\lambda) d\lambda$  and the total energy at all wavelengths is  $\int_0^\infty I(\lambda) d\lambda$ . Dividing one expression by the other and substituting for  $I(\lambda)$  from above, we get an expression for the efficiency  $\eta$  of the light bulb thus:

$$\eta = \frac{\int_{\lambda_1}^{\lambda_2} \lambda^{-5} / (e^{hc/\lambda k_B T} - 1) d\lambda}{\int_0^\infty \lambda^{-5} / (e^{hc/\lambda k_B T} - 1) d\lambda},$$

where the leading constants and the area  $A$  have canceled out. Making the substitution  $x = hc/\lambda k_B T$ , this can also be written as

$$\eta = \frac{\int_{hc/\lambda_1 k_B T}^{hc/\lambda_2 k_B T} x^3 / (e^x - 1) dx}{\int_0^\infty x^3 / (e^x - 1) dx} = \frac{15}{\pi^4} \int_{hc/\lambda_2 k_B T}^{hc/\lambda_1 k_B T} \frac{x^3}{e^x - 1} dx,$$

where we have made use of the known exact value of the integral in the denominator.

- (a) Write a Python function that takes a temperature  $T$  as its argument and calculates the value of  $\eta$  for that temperature from the formula above. The integral in the formula cannot be done analytically, but you can do it numerically using any method of your choice. (For instance, Gaussian quadrature with 100 sample points works fine.) Use your function to make a graph of  $\eta$  as a function of temperature between 300 K and 10 000 K. You should see that there is an intermediate temperature where the efficiency is a maximum.
  - (b) Calculate the temperature of maximum efficiency of the light bulb to within 1 K using golden ratio search. What efficiency does the bulb achieve at this temperature?
  - (c) Is it practical to run a tungsten-filament light bulb at the temperature you found? If not, why not?
- For full credit** turn in a printout of your final (golden-ratio search) program, your plot from part (a), and your results and discussion from parts (b) and (c).