Complex Systems 535/Physics 508: Homework 4

1. (i) The Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8 . . . They have the definitive property that each is the sum of the previous two. The generating function for the Fibonacci numbers is the power series whose coefficients are the Fibonacci numbers: $f(z) = z + z^2 + 2z^3 + 3z^4 + 5z^5 + \ldots$ Show that $f(z) = z/(1 - z - z^2)$.

   (ii) Consider the binomial distribution of an integer variable k:

   $$B_{nk}(p, q) = \binom{n}{k} p^k q^{n-k},$$

   where $q = 1 - p$. Derive a closed-form expression for the generating function $g(z) = \sum_{k=0}^{n} B_{nk} z^k$. Hence find the mean of the binomial distribution.

   (iii) A sequence of numbers a_k with $k = 1, 2, 3, \ldots$ satisfies the recurrence

   $$a_k = \begin{cases} 1 & \text{for } k = 1, \\ \sum_{j=1}^{k-1} a_j a_{k-j} & \text{for } k > 1. \end{cases}$$

   Show that the generating function $h(z) = \sum_{k=1}^{\infty} a_k z^k = \frac{1}{2}(1 - \sqrt{1 - 4z})$.

2. Suppose we have m nonnegative integers $k_1 \ldots k_m$ that are drawn independently, each from its own probability distribution: $p_1(k_1)$ is the distribution for k_1, $p_2(k_2)$ is the distribution for k_2, and so forth. Let $g_1(z)$, $g_2(z)$, etc. be the generating functions for these distributions.

   Let π_s be the probability that the sum $\sum_{i=1}^{m} k_m$ has the value s. Show that the generating function for π_s is $h(z) = \prod_i g_i(z)$. Hence, rederive the result we showed in class that the generating function for the sum of m independent draws from the same distribution $p(k)$ is the generating function for a single draw raised to the mth power.

3. Consider the configuration model with exponential degree distribution $p_k = (1 - e^{-\lambda})e^{-\lambda k}$ with $\lambda > 0$, so that the generating functions $g_0(z)$ and $g_1(z)$ are given by Eq. (13.130).

   (i) Show that the probability u in Eq. (13.91) satisfies the cubic equation $u^3 - 2e^{\lambda}u^2 + e^{2\lambda}u - (e^{\lambda} - 1)^2 = 0$.

   (ii) Noting that $u = 1$ is always a trivial solution of Eq. (13.91), show that the nontrivial solution corresponding to the existence of a giant component satisfies the quadratic equation $u^2 - (2e^{\lambda} - 1)u + (e^{\lambda} - 1)^2 = 0$, and hence that the size of the giant component, if there is one, is

   $$S = \tfrac{3}{2} - \sqrt{e^{\lambda} - \tfrac{3}{4}}.$$

   Sketch or plot the form of S as a function of λ.

   (iii) Show that the giant component exists only if $\lambda < \ln 3$.

4. **Extra credit:** Write a program in the computer language of your choice (or modify your program from last week) to generate a configuration model network in which there are vertices of degree 1 and 3 only and then calculate the size of the largest component.

   (i) Use your program to calculate the largest component for a network with $n = 10\,000$ nodes when $p_1 = 0.6$ and $p_3 = 0.4$ (and $p_k = 0$ for all other values of k).

   (ii) Modify your program to make a graph of the size of the largest component for values of p_1 from 0 to 1 in steps of 0.01. Hence estimate the value of p_1 for the phase transition at which the giant component disappears.