Complex Systems 535/Physics 508: Homework 5

1. Consider a configuration model in which every vertex has the same degree k.

   (i) What is the degree distribution p_k? What are the generating functions g_0 and g_1 for the degree distribution and the excess degree distribution?

   (ii) Show that the probability of a node belonging to the giant component is 1 for all $k \geq 3$, i.e., that the giant component is the size of the entire network, at least in the limit of large network size.

   (iii) What happens when $k = 1$?

   (iv) **Extra credit:** What happens when $k = 2$?

2. (i) Let us model the Internet as a configuration model with a perfect power-law degree distribution $p_k \sim k^{-\alpha}$, with $\alpha \simeq 2.5$ and $k \geq 1$. Write down the fundamental generating functions g_0 and g_1.

   (ii) Hence estimate what fraction of the nodes on the Internet you expect to be functional at any one time (where functional means they belong to the largest component).

3. Consider a model of a growing directed network similar to Price's model described in Section 14.1, but without preferential attachment. That is, vertices are added one by one to the growing network and each has c outgoing edges, but those edges now attach to existing vertices uniformly at random, without regard for degrees or any other vertex property.

   (i) Derive master equations, the equivalent of Eqs. (14.7) and (14.8), that govern the distribution of in-degrees q in the limit of large network size.

   (ii) Hence show that in the limit of large size the in-degrees have an exponential distribution $p_q = C\mathrm{e}^{-\lambda q}$ with $\lambda = \ln(1 + 1/c)$.

4. Consider a model network similar to the model of Barabási and Albert described in Section 14.2, in which undirected edges are added between vertices according to a preferential attachment rule, but suppose that the network does not grow—it starts off with a given number n of vertices and neither gains nor loses any vertices thereafter. In this model, starting with an initial network of n vertices and some specified arrangement of edges, we add at each step one undirected edge between two vertices, both of which are chosen at random in direct proportion to degree k. Let $p_k(m)$ be the fraction of vertices with degree k when the network has m edges in total.

   (i) Show that, when the network has m edges, the probability that the next edge added will attach to vertex i is k_i/m.

   (ii) Write down a master equation giving $p_k(m+1)$ in terms of $p_{k-1}(m)$ and $p_k(m)$. Give the equation for the special case of $k = 0$ also.

   (iii) Eliminate m from the master equation in favor of the mean degree $c = 2m/n$ and take the limit $n \to \infty$ with c held constant to show that $p_k(c)$ satisfies the differential equation

$$c\frac{\mathrm{d}p_k}{\mathrm{d}c} = (k-1)p_{k-1} - kp_k.$$

(iv) Define a generating function $g(c,z) = \sum_{k=0}^{\infty} p_k(c)\, z^k$ and show that it satisfies the partial differential equation

$$c\frac{\partial g}{\partial c} + z(1-z)\frac{\partial g}{\partial z} = 0.$$

(v) Show that $g(c,z) = f(c - c/z)$ is a solution of this differential equation, where $f(x)$ is any differentiable function of x.

(vi) The particular choice of f depends on the initial conditions on the network. Suppose the network starts off in a state where every vertex has degree one, which means $c = 1$ and $g(1,z) = z$. Find the function f that corresponds to this initial condition and hence find $g(c,z)$ for all values of c and z.

(vii) Show that, for this solution, the degree distribution as a function of c takes the form

$$p_k(c) = \frac{(c-1)^{k-1}}{c^k},$$

except for $k = 0$, for which $p_0(c) = 0$ for all c.

Note that the degree distributions in both this model and the model of question 3 decay exponentially in k, implying that neither preferential attachment nor network growth alone can account for a power-law degree distribution. One must have both growth and preferential attachment to get a power law.