Physics 411: Homework 4

1. **Numerical differentiation:** Create a user-defined function `f(x)` that returns the value $1 + \frac{1}{2}\tanh 2x$, then use a central difference to calculate the derivative of the function in the range $-2 \leq x \leq 2$. Calculate an analytic formula for the derivative and make a graph with your numerical result and the analytic answer on the same plot. It may help to plot the exact answer as lines and the numerical one as dots. (Hint: In Python the tanh function is found in the `math` package, and it's called simply `tanh`.)

   ☑ **For full credit** turn in a copy of your program and the plot it produces.

2. **An alternative integration rule:** Rearranging Eq. (5.19) into a slightly more conventional form, we have:

$$\int_a^b f(x)\,\mathrm{d}x = h\left[\tfrac{1}{2}f(a) + \tfrac{1}{2}f(b) + \sum_{k=1}^{N-1} f(a+kh)\right] + \tfrac{1}{12}h^2\left[f'(a) - f'(b)\right] + \mathrm{O}(h^4).$$

   This result gives a value for the integral on the left which has an error of order h^4—a factor of h^2 better than the error on the trapezoidal rule and as good as Simpson's rule. We can use this formula as a new rule for evaluating integrals, distinct from any of the others we have seen so far. We might call it the "Euler–Maclaurin rule" after the Euler–Maclaurin formula on which it is based.

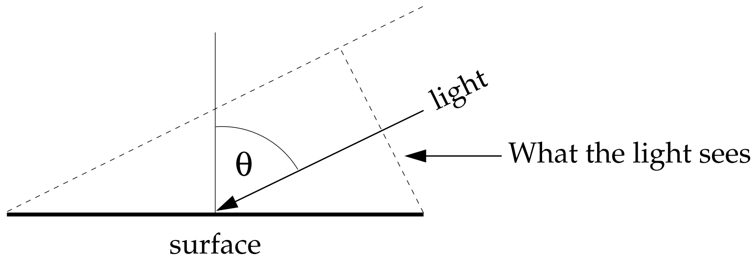
   (a) Write a program to calculate the value of the integral $\int_0^2 (x^4 - 2x + 1)\,\mathrm{d}x$ using this formula. (This is the same integral that we studied in several previous problems, whose true value is 4.4.) The order-h term in the formula is just the ordinary trapezoidal rule; the h^2 term involves the derivatives $f'(a)$ and $f'(b)$, which you should evaluate using central differences, centered on a and b respectively. Note that the size of the interval you use for calculating the central differences does not have to equal the value of h used in the trapezoidal rule part of the calculation. An interval of about 10^{-5} gives good values for the central differences.

   Use your program to evaluate the integral with $N = 10$ slices and compare the accuracy of the result with that obtained from the trapezoidal rule alone with the same number of slices.

   (b) Good though it is, this integration method is not much used in practice. Suggest a reason why not.

   ☑ **For full credit** turn in a copy of your program, a printout of it in action showing the result it produces, and your answer to the question in part (b).

3. **Image processing and the STM:** When light strikes a surface, the amount falling per unit area depends not only on the intensity of the light, but also on the angle of incidence. If the light makes an angle θ to the normal, it only "sees" $\cos\theta$ of area per unit of actual area on the surface:



So the intensity of illumination is $a\cos\theta$, if a is the raw intensity of the light. This simple physical law is a central element of 3D computer graphics. It allows us to calculate how light falls on three-dimensional objects and hence how they will look when illuminated from various angles.

Suppose, for instance, that we are looking down on the Earth from above and we see mountains. We know the height of the mountains $w(x,y)$ as a function of position in the plane, so the equation for the Earth's surface is simply $z = w(x,y)$, or equivalently $w(x,y) - z = 0$, and the normal vector $\mathbf{v}$ to the surface is given by the gradient of $w(x,y) - z$ thus:

$$\mathbf{v} = \nabla[w(x,y) - z] = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} [w(x,y) - z] = \begin{pmatrix} \partial w/\partial x \\ \partial w/\partial y \\ -1 \end{pmatrix}.$$

Now suppose we have incident light represented by a vector $\mathbf{a}$ with magnitude equal to the intensity of the light. Then the dot product of the vectors $\mathbf{a}$ and $\mathbf{v}$ is

$$\mathbf{a}\cdot\mathbf{v} = |\mathbf{a}|\,|\mathbf{v}|\cos\theta,$$

where θ is the angle between the vectors, and the intensity of illumination of the surface of the mountains is

$$I = |\mathbf{a}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{a_x(\partial w/\partial x) + a_y(\partial w/\partial y) - a_z}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

Let's take a simple case where the light is shining horizontally with unit intensity, along a line an angle ϕ counter-clockwise from the east-west axis, so that $\mathbf{a} = (\cos\phi, \sin\phi, 0)$. Then our intensity of illumination simplifies to

$$I = \frac{\cos\phi\,(\partial w/\partial x) + \sin\phi\,(\partial w/\partial y)}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

If we can calculate the derivatives of the height $w(x,y)$ and we know ϕ we can calculate the intensity at any point.

(a) In the on-line resources you'll find a file called `altitude.txt`, which contains the altitude $w(x,y)$ in meters above sea level (or depth below sea level) of the surface of the Earth, measured on a grid of points (x,y). Write a program that reads this file and stores the data in an array. Then calculate the derivatives $\partial w/\partial x$ and $\partial w/\partial y$ at each grid point. Explain what method you used to calculate them and why. (Hint: You'll probably have to use more than one method to get every grid point, because awkward things happen at the edges of the grid.) To calculate the derivatives you'll need to know the value of h, the distance in meters between grid points, which is about 30 000 m in this case. (It's actually not precisely constant because we are representing the spherical Earth on a flat map, but $h = 30\,000$ m will give reasonable results.)

(b) Now, using your values for the derivatives, calculate the intensity for each grid point, with $\phi = 45°$, and make a density plot of the resulting values in which the brightness of each dot depends on the corresponding intensity value. If you get it working right, the plot should look like a relief map of the world—you should be able to see the continents and mountain ranges in 3D. (Common problems include a map that is upside-down or sideways, or a relief map that is "inside-out," meaning the high regions look low and *vice versa*. Work with the details of your program until you get a map that looks right to you.)

(c) There is another file in the on-line resources called `stm.txt`, which contains a grid of values from scanning tunneling microscope (STM) measurements of the (111) surface of silicon—the same file that you used in Homework 2. The grid of values in this file represents the height of the silicon surface as a function of position. Modify the program you just wrote to visualize the STM data and hence create a 3D picture of what the silicon surface looks like. The value of h for the derivatives in this case is around $h = 2.5$ (in arbitrary units).

☑ **For full credit** turn in your explanation from part (a), copies of your two density plots, from parts (b) and (c), and your final program from part (c).

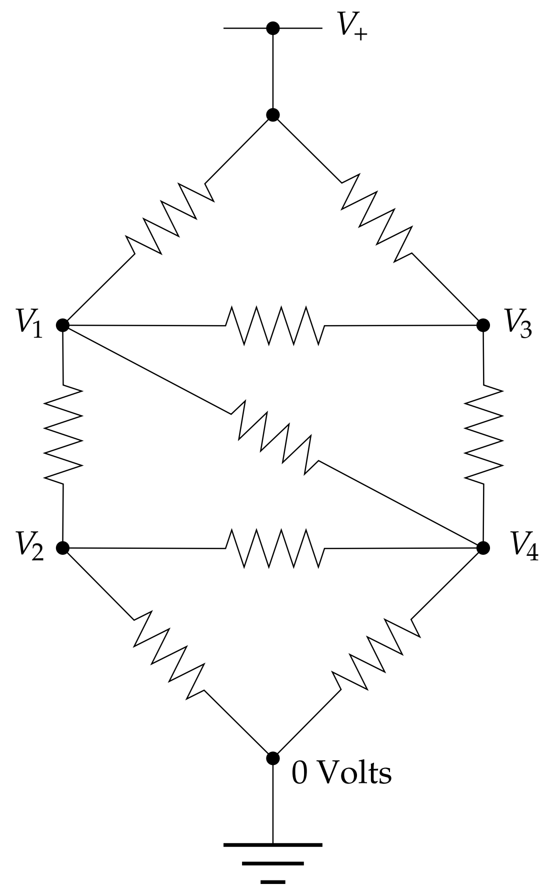
4. **Pivoting:**

(a) Modify the program `gausselim.py` in Example 6.1 (which is also in the on-line resources) to incorporate partial pivoting (or you can write your own program from scratch if you prefer). Run your program and demonstrate that it gives the same answers as the original program when applied to Eq. (6.1).

(b) Modify the program to solve the equations in (6.17) and show that it can find the solution to those as well, even though Gaussian elimination without pivoting fails.

☑ **For full credit** turn in a copy of your final program and a printout of it in action, showing the results it gives for part (b).

5. **A circuit of resistors:** Consider the following circuit of resistors:



All the resistors have the same resistance R. The power rail at the top is at voltage $V_+ = 5\,\text{V}$. The question is, what are the other four voltages, V_1 to V_4?

To answer this question we use Ohm's law and the Kirchhoff current law, which says that the total net current flow out of (or into) any junction in a circuit must be zero. Thus for the junction at voltage V_1, for instance, we have

$$\frac{V_1 - V_2}{R} + \frac{V_1 - V_3}{R} + \frac{V_1 - V_4}{R} + \frac{V_1 - V_+}{R} = 0,$$

or equivalently

$$4V_1 - V_2 - V_3 - V_4 = V_+.$$

(a) Derive similar equations for the other three junctions with unknown voltages.

(b) Write a program to solve the four resulting equations using Gaussian elimination and hence find the four voltages (or you can modify a program you already have, such as the one you wrote for problem 4).

☑ **For full credit** turn in your derivations from part (a), a copy of your program, and a printout of it in action showing the results it gives for the voltages.