Physics 390: Homework 9

For full credit, show all your working.

1. **White dwarf stars:** Living stars such as our Sun hold their shape against gravity because of the ordinary (but very high) hydrodynamic pressure created by their heat. When stars die and stop shining, however, they become cool and collapse under their own weight to form white dwarf stars. White dwarfs are held up not by conventional pressure but by the degeneracy pressure of the Fermi gas formed by their electrons. In the calculations below, assume that the electrons can be treated as a non-interacting Fermi gas of the kind discussed in class, filling the entire volume of the star.

   (a) Consider a spherical white dwarf star of mass M, radius R, and uniform density. White dwarfs are made of carbon and oxygen, so essentially all of their mass comes from carbon and oxygen nuclei, in the form of equal numbers of protons (mass m_p and charge $+e$) and neutrons (the same mass and zero charge). (The electrons are so much lighter that they make almost no contribution to the mass.) Given that our star is electrically neutral, give an expression for the number of electrons it contains. Hence write an expression for the number density ρ of electrons (the number of electrons per unit volume).

   (b) The total kinetic energy of a non-interacting Fermi gas at $T = 0$ is given by

   $$E_e = \int_0^{E_F} E\, g(E)\, \mathrm{d}E,$$

   where E_F is the Fermi energy (also sometimes denoted μ). Using the formulas for the Fermi energy and the density of states $g(E)$ from the class (and remembering the factor of 2 because of the spin states), show that $E_e = \frac{3}{5} N E_F$.

   (c) Hence, assuming the star to be at temperature $T = 0$, show that the total kinetic energy of the electrons in the star is

   $$E_e = \frac{3\hbar^2}{10 m_e R^2} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{2m_p}\right)^{5/3},$$

   where m_e is the mass of the electron.

   (d) It can be shown by simple mechanics that the gravitational potential energy of the star is

   $$E_g = -\frac{3GM^2}{5R},$$

   where G is Newton's gravitational constant. By minimizing the total energy $E = E_g + E_e$ of the star, show that the radius of the star depends on its mass as

   $$R = \frac{3\hbar^2}{8Gm_e} \left(\frac{3\pi^2}{m_p^5}\right)^{1/3} M^{-1/3}.$$

(e) When the Sun finally dies and becomes a white dwarf, what will its radius be? How does this compare to its current radius?

2. **Stable nuclei:** Equation (11-14) in the handout gives the semi-empirical formula for the total energy of a nucleus to be:

$$E = Zm_pc^2 + Nm_nc^2 - [a_1A - a_2A^{2/3} - a_3Z^2A^{-1/3} - a_4(A-2Z)^2/A + a_5A^{-1/2}]c^2.$$

(a) Show that, for a given value of A, the minimum energy occurs when

$$Z = \tfrac{1}{2}A\,\frac{4a_4 + m_n - m_p}{4a_4 + a_3A^{2/3}}.$$

(b) Using the values of the constants given in the handout, determine the most stable nuclei with $A = 29$, 59, and 78. How do your results compare with the data in Appendix A of the book?

(c) **Extra credit:** Make a computer plot of the stable line from the formula above on N and Z axes and on the same plot show the known stable nuclides. A list of the stable nuclides can be found at `umich.edu/~mejn/courses/2015/phys390/stable.txt` and also on the course web page.

3. **Radioactive decay:**

(a) When we say that the atomic mass of ^{226}Ra, for example, is 226.0254, what units is that measured in? In other words, that's 226.0254 of what?

(b) ^{226}Ra is an α emitter. Calculate the energy of the α particles from ^{226}Ra, assuming the kinetic energy of the recoiling nucleus to be negligible.

(c) ^{228}Ra is a β^- emitter. Calculate the *maximum* energy an electron emitted by ^{228}Ra can have, again neglecting recoil. (Hint: Note that the atomic masses listed in the book really are *atomic* masses, including the electrons, not just the masses of the nuclei.)

4. **The Sun:** The Sun is powered by a nuclear fusion reaction called the *pp chain* in which, through a series of processes, four protons (in the form of hydrogen nuclei) are fused together into a helium, with the accompanying emission of two positrons and a bunch of neutrinos and γ rays. Most of the energy emerges in the form of γ radiation.

(a) From the masses of the proton and the helium nucleus and the total luminosity of the Sun, make an estimate of the number of kilograms of hydrogen the Sun converts into helium per second.

(b) Assuming that most of the mass of the Sun is in the form of hydrogen, about how many years will the Sun burn for before it collapses and becomes a white dwarf?