Physics 390: The quantum simple harmonic oscillator

The energy eigenstates of the simple harmonic oscillator satisfy

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \tfrac{1}{2}m\omega^2x^2\psi = E\psi. \tag{1}$$

Consider the new function ψ_- defined by

$$\psi_- = \frac{\hbar}{m\omega}\frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi. \tag{2}$$

Multiplying by $-\frac{1}{2}\hbar\omega$ and differentiating we find that

$$\begin{aligned}
-\tfrac{1}{2}\hbar\omega\frac{\mathrm{d}\psi_-}{\mathrm{d}x} &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \tfrac{1}{2}\hbar\omega\psi - \tfrac{1}{2}\hbar\omega x\frac{\mathrm{d}\psi}{\mathrm{d}x} \\
&= E\psi - \tfrac{1}{2}m\omega^2x^2\psi - \tfrac{1}{2}\hbar\omega\psi - \tfrac{1}{2}\hbar\omega x\frac{\mathrm{d}\psi}{\mathrm{d}x} \\
&= (E - \tfrac{1}{2}\hbar\omega)\psi - \tfrac{1}{2}m\omega^2x\left(\frac{\hbar}{m\omega}\frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi\right) \\
&= (E - \tfrac{1}{2}\hbar\omega)\psi - \tfrac{1}{2}m\omega^2x\psi_-\,,
\end{aligned} \tag{3}$$

where we have made use of Eqs. (1) and (2) in the second and fourth lines respectively. Now multiplying by $\hbar/m\omega$ and differentiating again we get

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_-}{\mathrm{d}x^2} &= \frac{\hbar}{m\omega}(E - \tfrac{1}{2}\hbar\omega)\frac{\mathrm{d}\psi}{\mathrm{d}x} - \tfrac{1}{2}\hbar\omega\psi_- - \tfrac{1}{2}\hbar\omega x\frac{\mathrm{d}\psi_-}{\mathrm{d}x} \\
&= \frac{\hbar}{m\omega}(E - \tfrac{1}{2}\hbar\omega)\frac{\mathrm{d}\psi}{\mathrm{d}x} - \tfrac{1}{2}\hbar\omega\psi_- + (E - \tfrac{1}{2}\hbar\omega)x\psi - \tfrac{1}{2}m\omega^2x^2\psi_-\,,
\end{aligned} \tag{4}$$

where we have used Eq. (3) to eliminate $\mathrm{d}\psi_-/\mathrm{d}x$. Rearranging this expression and collecting terms, we find that

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_-}{\mathrm{d}x^2} + \tfrac{1}{2}m\omega^2x^2\psi_- = (E - \tfrac{1}{2}\hbar\omega)\underbrace{\left[\frac{\hbar}{m\omega}\frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi\right]}_{\psi_-} - \tfrac{1}{2}\hbar\omega\psi_-\,. \tag{5}$$

As noted, the quantity in square brackets is none other than ψ_- and hence we find that

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_-}{\mathrm{d}x^2} + \tfrac{1}{2}m\omega^2x^2\psi_- = (E - \hbar\omega)\psi_-\,. \tag{6}$$

But this is simply the Schrödinger equation again. It tells us that if the original wavefunction ψ was a solution of the Schrödinger equation, then so too is ψ_- defined by Eq. (2), but with a different energy $E-\hbar\omega$, exactly $\hbar\omega$ lower than the original energy.

Thus if we can find one solution to the Schrödinger equation, with any energy E, then we can immediately find another one with energy $E-\hbar\omega$. But now we can repeat the process and hence find a state with an energy $\hbar\omega$ lower still, and so forth. In this way we can get a whole "ladder" of solutions to the Schrödinger equation with energies $\hbar\omega$ apart. We simply keep applying Eq. (2). (We can also go in the opposite direction and increase energy by $\hbar\omega$, extending the ladder in the upward direction as well—see this week's homework set.)

There is, however, a catch. The energy cannot go on decreasing forever: the energy of the simple harmonic oscillator is a sum of nonnegative quantities, so it is itself nonnegative. But how can this be, when we have just shown that we can go on decreasing the energy by steps of $\hbar\omega$ for as long as we like? The answer is that at some point there must be a solution ψ of the equation for which the trick above does not work and Eq. (2) does not generate another solution with lower energy. How could this happen mathematically? Well, it happens if

$$\frac{\hbar}{m\omega}\frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi = 0. \tag{7}$$

In this case, ψ_- still technically satisfies Eq. (6) (because both sides will be zero), but we don't get a new solution with lower energy.

If we multiply Eq. (7) by $\hbar/m\omega$ and differentiate, we get

$$\frac{\hbar^2}{m^2\omega^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{\hbar}{m\omega}\psi + \frac{\hbar}{m\omega}x\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\hbar^2}{m^2\omega^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{\hbar}{m\omega}\psi - x^2\psi = 0, \tag{8}$$

where we have used Eq. (7) to eliminate $\mathrm{d}\psi/\mathrm{d}x$. Now we multiply this equation by $-\frac{1}{2}m\omega^2$ and rearrange to get

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \tfrac{1}{2}m\omega^2 x^2\psi = \tfrac{1}{2}\hbar\omega\psi. \tag{9}$$

This is just the Schrödinger equation again, and so we see that this wavefunction, the wavefunction of the state at the bottom of the ladder, which is the *ground state* of the simple harmonic oscillator, has energy $E = \frac{1}{2}\hbar\omega$.

Thus, by a process of deduction, we conclude that the energy levels of the simple harmonic oscillator start at $\frac{1}{2}\hbar\omega$ and go up in steps of $\hbar\omega$, so that the nth energy level has energy

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega. \tag{10}$$

As a final trick, we can also solve for the actual wave function of the ground state, by solving the differential equation (7). Separating the variables and integrating gives

$$\int \frac{\mathrm{d}\psi}{\psi} = -\frac{m\omega}{\hbar}\int x\,\mathrm{d}x, \tag{11}$$

or

$$\psi(x) = A\exp(-m\omega x^2/2\hbar), \tag{12}$$

where A is a normalization constant. You can check that this agrees with the answer claimed in the book (Eq. (6-58) in Tipler & Llewellyn).