Physics 390: The quantum simple harmonic oscillator

The energy eigenstates of the simple harmonic oscillator satisfy $H\psi = E\psi$ with

$$H = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2x^2, \tag{1}$$

where p is the quantum momentum operator

$$p = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}. \tag{2}$$

To make things a bit simpler, we change variables to $y = x\sqrt{m\omega/\hbar}$, so that

$$H = -\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{m\omega}{2}\frac{\hbar}{m\omega}y^2 = -\frac{\hbar\omega}{2}\biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2\biggr]. \tag{3}$$

Then the Schrödinger equation becomes

$$\biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2\biggr]\psi = -\frac{2E}{\hbar\omega}\psi. \tag{4}$$

Now here's the trick. Notice that for any function $f(y)$

$$\begin{aligned}\biggl[\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr) - 1\biggr]f &= \Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr)\Bigl(\frac{\mathrm{d}f}{\mathrm{d}y} + yf\Bigr) - f \\ &= \frac{\mathrm{d}^2f}{\mathrm{d}y^2} + f + y\frac{\mathrm{d}f}{\mathrm{d}y} - y\frac{\mathrm{d}f}{\mathrm{d}y} - y^2f - f \\ &= \biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2\biggr]f \end{aligned} \tag{5}$$

Similarly you can show that

$$\biggl[\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr) + 1\biggr]f = \biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2\biggr]f. \tag{6}$$

Using Eq. (5) we can write Eq. (4) as

$$\biggl[\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr) - 1\biggr]\psi = -\frac{2E}{\hbar\omega}\psi, \tag{7}$$

or equivalently

$$\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi = \Bigl(1 - \frac{2E}{\hbar\omega}\Bigr)\psi. \tag{8}$$

Now we operate on both sides with the operator $\mathrm{d}/\mathrm{d}y + y$ to get

$$\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} - y\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi = \Bigl(1 - \frac{2E}{\hbar\omega}\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi, \tag{9}$$

and then make use of Eq. (6) to write this as

$$\biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2 - 1\biggr]\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi = \Bigl(1 - \frac{2E}{\hbar\omega}\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi, \tag{10}$$

or

$$\biggl[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2\biggr]\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi = \Bigl(2 - \frac{2E}{\hbar\omega}\Bigr)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi. \tag{11}$$

Now we multiply both sides by $-\frac{1}{2}\hbar\omega$ and make use of Eq. (3) again to get

$$H\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi = (E - \hbar\omega)\Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi. \tag{12}$$

It's a lot of algebra, but the final result is interesting. If we define a new function ψ' by

$$\psi' = \Bigl(\frac{\mathrm{d}}{\mathrm{d}y} + y\Bigr)\psi, \tag{13}$$

then Eq. (12) can be written as

$$H\psi' = (E - \hbar\omega)\psi'. \tag{14}$$

In other words, ψ' is a solution of the Schrödinger equation, but with an energy that's $\hbar\omega$ less than the energy for ψ.

What this means is that if we can find one solution to the Schrödinger equation, with any energy E, then we can immediately find another one with energy $E-\hbar\omega$ by applying the operator $\mathrm{d}/\mathrm{d}y+y$. This operator is called a "lowering operator"—it generates a new solution of lower energy from the old one.

But now we can repeat the process and apply the same operator again to ψ' and get ψ'', which has energy $\hbar\omega$ lower still, and so forth. In this way we can get a whole "ladder" of solutions to the Schrödinger equation with energies $\hbar\omega$ apart. In theory we can also go in the opposite direction and increase energy by $\hbar\omega$, extending the ladder in the upward direction as well.

There is, however, a catch. The energy cannot go on decreasing forever: the energy of the simple harmonic oscillator, Eq. (1), is a sum of nonnegative quantities, so it is itself nonnegative. But how can this be, when we have just shown that we can go on decreasing the energy by steps of $\hbar\omega$ for as long as we like? The answer is that at some point there must be a solution ψ of the equation for which the trick above does not work and the operator $\mathrm{d}/\mathrm{d}y + y$ does *not* generate another solution with lower energy. How could this happen mathematically? Well, it happens if

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} + y\right)\psi = 0. \tag{15}$$

Technically, such a function still satisfies Eq. (12) (because both sides will be zero), but we don't get a new solution with lower energy.

If we take this equation and operate on both sides with $\mathrm{d}/\mathrm{d}y - y$ and use Eq. (5) again, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - y\right)\left(\frac{\mathrm{d}}{\mathrm{d}y} + y\right)\psi = \left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - y^2 + 1\right]\psi = 0. \tag{16}$$

Then multiplying both sides by $-\frac{1}{2}\hbar\omega$ and making use of (3) we get

$$H\psi = \tfrac{1}{2}\hbar\omega\psi. \tag{17}$$

Comparing with the standard Schrödinger equation $H\psi = E\psi$, we see that the state at the bottom of the ladder—the *ground state* of the simple harmonic oscillator—has energy $E = \frac{1}{2}\hbar\omega$.

Thus, by a process of deduction, we conclude that the energy levels of the simple harmonic oscillator start at $\frac{1}{2}\hbar\omega$ and go up in steps of $\hbar\omega$, so that the nth energy level has energy

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega. \tag{18}$$

As a final trick, we can also solve for the actual wave function of the ground state, by solving the differential equation (15). Separating the variables and integrating gives

$$\int \frac{\mathrm{d}\psi}{\psi} = -\int y\,\mathrm{d}y, \tag{19}$$

or

$$\psi = A\exp(-y^2/2) = A\exp(-m\omega x^2/2\hbar), \tag{20}$$

where A is a normalization constant. You can check that this agrees with the answer claimed in the book (Eq. (6-58) in Tipler & Llewellyn).