Proposition 3: Individual Round

Name: ____________________

Team ID: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 50 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Proposition 3: Individual Round

1. Alex is a lazy student. His teacher, Stephen, is going to give him a test with 500 questions, and Alex must answer 10 of them. However, Stephen is nice and gives him a list of 2020 questions beforehand, guaranteeing that the questions on the test will be picked from that list. Being a lazy student (but a lazy A-student!), Alex wants to ensure that he can answer 10 questions correctly, no matter what 500 questions appear on the test. What is the minimum number of the 2020 questions Alex needs to be able to answer correctly to ensure that he can do this?

1. ______________

2. Determine the sum of the coefficients in the expansion of $(3x - 5y)^6$.

2. ______________

3. You are given a basket filled with balls marked with numbers. There are 2020 balls marked "1", 2019 balls marked "2", 2018 balls marked "3", and so on, until 1 ball marked "2020". What is the size of the largest set of balls in the basket such that no two are marked with numbers that sum to a multiple of 4?

3. ______________

4. In triangle $\triangle ABC$, $AB + AC = 6$, and the area of $\triangle ABC$ is 4. Find the maximum possible value of $\cos(\angle A)$.

4. ______________

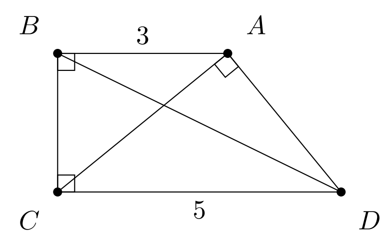
5. 2020 people stand in a circle, all holding their right hand out in front of them, towards the center of the circle. At once, they all instantly turn to face either the person on the left or the person on their right, each with 50% probability. If two people are now facing each other, they will shake hands. What is the expected number of handshakes that occur?

5. ______________

6. Consider a trapezoid $ABCD$, where

   - $\overline{AB}$ is perpendicular to $\overline{BC}$
   - $\overline{BC}$ is perpendicular to $\overline{CD}$
   - $\overline{AC}$ is perpendicular to $\overline{AD}$
   - The length of $\overline{AB}$ is 3 and the length of $\overline{CD}$ is 5.

   What is the product of the lengths of $\overline{AC}$ and $\overline{BD}$?

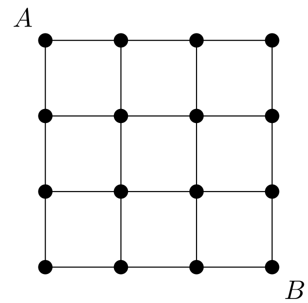


6. ______________

7. Determine the smallest positive integer n such that $n^{22} - 1$ is divisible by 23, but $n^k - 1$ is not divisible by 23 for any positive integer $k < 22$.

7. ______________

8. Noah walks from point A to point B on the grid below.



He tries to take the shortest path (that is, walk only rightwards and downwards). Unfortunately, Noah has an imperfect sense of direction, so he walks *exactly once* in the wrong direction (that is, either leftwards or upwards). He always stays in the depicted grid. How many possible paths could Noah have taken? (Note that he only reaches point B once).

8. ______________

9. The value of $2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right)$ can be expressed in the form $\frac{a}{b}\pi$, where a and b are relatively prime positive integers. Find $a + b$.

9. ______________

10. Triangle $\triangle ABC$ has dimensions $BC = a$, $CA = b$, $AB = c$. If a, b, c forms a geometric progression, and $\sin(\angle B - \angle A), \sin(\angle A), \sin(\angle C)$ forms a arithmetic progression, determine $\cos(\angle B)$.

10. ______________