Proposition 3: Individual Round

Name: ____

Team ID: ____

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 50 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Proposition 3: Individual Round

1. Alex is a lazy student. His teacher, Stephen, is going to give him a test with 500 questions, and Alex must answer 10 of them. However, Stephen is nice and gives him a list of 2020 questions beforehand, guaranteeing that the questions on the test will be picked from that list. Being a lazy student (but a lazy A-student!), Alex wants to ensure that he can answer 10 questions correctly, no matter what 500 questions appear on the test. What is the minimum number of the 2020 questions Alex needs to be able to answer correctly to ensure that he can do this?

1. 1530

Solution: In the worst case, Alex knows how to solve only 10 of the presented 500 questions and also knows how to solve all of the other $2020 - 500 = 1520$ questions. So, Alex must know how to solve at least $10 + 1520 = 1530$ questions.

2. Determine the sum of the coefficients in the expansion of $(3x - 5y)^6$.

2. 64

Solution: Note that the sum of the coefficients is the value of $(3x - 5y)^6$ when $x = 1$ and $y = 1$. Evaluating, we have $(3x - 5y)^6 = (3 - 5)^6 = (-2)^6 = 64$.

3. You are given a basket filled with balls marked with numbers. There are 2020 balls marked "1", 2019 balls marked "2", 2018 balls marked "3", and so on, until 1 ball marked "2020". What is the size of the largest set of balls in the basket such that no two are marked with numbers that sum to a multiple of 4?

3. 511062

Solution: Note that if we pick a ball marked with a number that is 1 more than a multiple of 4, then we cannot pick a ball marked with a number that is 1 less than a multiple of 4. The opposite also holds. In addition, we can pick exactly one ball marked with a multiple of 4, and we can pick exactly one ball marked with a number that is 2 more than a multiple of 4. These last two choices do not affect the choice of balls marked with odd numbers.

Putting these together, it follows that we should pick 2 even balls, and either all of the balls marked with numbers that are 1 more than a multiple of 4 or all balls marked with numbers that are 1 less than a multiple of 4. It is clear that the former is larger, so we pick a total number of balls equal to

$$\begin{aligned} 2 + 2020 + 2016 + 2012 + \cdots + 4 &= 2 + \sum_{i=1}^{505} 4i \\ &= 2 + 2(505)(506) \\ &= 511062. \end{aligned}$$

4. In triangle $\triangle ABC$, $AB + AC = 6$, and the area of $\triangle ABC$ is 4. Find the maximum possible value of $\cos(\angle A)$.

4. $\frac{\sqrt{17}}{9}$

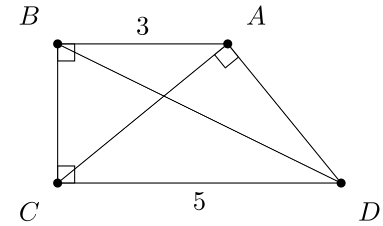
Solution: Recall that the area of $\triangle ABC$ is $\frac{1}{2}(AB)(AC)\sin(A)$. Since $AB + AC = 6$ is constant, the area is maximized when $AB = \frac{6}{2} = 3 = AC$. This means that $\sin(A)$ is minimized when $\frac{1}{2} \cdot 3 \cdot 3\sin(A) = 4$. Solving, the minimal value of $\sin(A)$ is $\frac{8}{9}$. It follows that the maximal value of $\cos(A)$ is $\sqrt{1 - \left(\frac{8}{9}\right)^2} = \frac{\sqrt{17}}{9}$.

5. 2020 people stand in a circle, all holding their right hand out in front of them, towards the center of the circle. At once, they all instantly turn to face either the person on the left or the person on their right, each with 50% probability. If two people are now facing each other, they will shake hands. What is the expected number of handshakes that occur?

5. 505

Solution: Consider any pair of adjacent people. The probability that they shake hands is $\frac{1}{4}$. There are 2020 such pairs. By linearity of expectation, the expected number of handshakes is $2020 \cdot \frac{1}{4} = 505$.

6. Consider a trapezoid $ABCD$, where

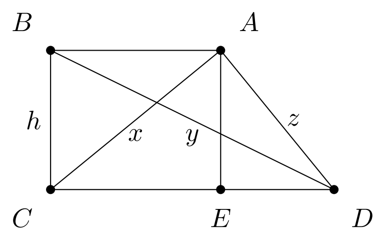


- $\overline{AB}$ is perpendicular to $\overline{BC}$
- $\overline{BC}$ is perpendicular to $\overline{CD}$
- $\overline{AC}$ is perpendicular to $\overline{AD}$
- The length of $\overline{AB}$ is 3 and the length of $\overline{CD}$ is 5.

What is the product of the lengths of $\overline{AC}$ and $\overline{BD}$?

6. $\sqrt{465}$

Solution: We modify the diagram to look like this:



Applying the Pythagorean theorem four times, we have

$$h^2 + 25 = y^2$$
$$h^2 + 9 = x^2$$
$$h^2 + 4 = z^2$$
$$x^2 + z^2 = 25.$$

This is a system of four equations with four unknowns (h^2, x^2, y^2, and z^2), so we can solve it. We find that $h^2 = 6$, $x^2 = 31$, $y^2 = 15$, and $z^2 = 10$. We want to find x^2y^2, which is just $31 \cdot 15 = 465$.

7. Determine the smallest positive integer n such that $n^{22}-1$ is divisible by 23, but n^k-1 is not divisible by 23 for any positive integer $k<22$.

7. 5

Solution: Since $n^{22}-1$ is divisible by 23, we know that $n^{22}\equiv 1 \pmod{23}$. Let p be the smallest nonzero integer for which $n^p\equiv 1 \pmod{23}$. Note that p exists since 22 is one such integer, so $p\le 22$. In particular, p must divide 22, and so p could possibly be 1, 2, 11, or 22. By Fermat's Little Theorem, $n^{22}\equiv 1 \pmod{23}$ for all integer n, so it suffices to check integers for which n, n^2, and n^{11} are not equivalent to 1 modulo 23.

Clearly if $n=1$, then $n\equiv 1 \pmod{23}$. We check for $n=2$. We compute

$$\begin{aligned}
2^1 &\equiv 2 \pmod{23}\\
2^2 &\equiv 4 \pmod{23}\\
2^4 &\equiv 16 \pmod{23}\\
2^8 &\equiv 3 \pmod{23}\\
2^{11} &\equiv 2^8 2^2 2^1 \equiv 3\cdot 4\cdot 2\equiv 24\equiv 1 \pmod{23}
\end{aligned}$$

Since $2^{11}\equiv 1 \pmod{23}$, 2 does not work. We then check $n=3$.

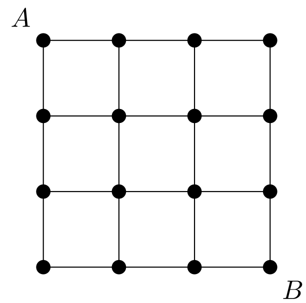
$$\begin{aligned}
3^1 &\equiv 3 \pmod{23}\\
3^2 &\equiv 9 \pmod{23}\\
3^4 &\equiv 12 \pmod{23}\\
3^8 &\equiv 6 \pmod{23}\\
3^{11} &\equiv 3^8 3^2 3^1 \equiv 6\cdot 9\cdot 3\equiv 162\equiv 1 \pmod{23}
\end{aligned}$$

Since $3^{11}\equiv 1 \pmod{23}$, 3 does not work. We know that $4^{11}=2^{22}\equiv 1 \pmod{23}$, so $n=4$ does not work. We check for $n=5$.

$$\begin{aligned}
5^1 &\equiv 5 \pmod{23}\\
5^2 &\equiv 2 \pmod{23}\\
5^4 &\equiv 4 \pmod{23}\\
5^8 &\equiv 16 \pmod{23}\\
5^{11} &\equiv 5^8 5^2 5^1 \equiv 16\cdot 2\cdot 5\equiv 160\equiv -1 \pmod{23}
\end{aligned}$$

Therefore, 5 is the minimal positive integer which works.

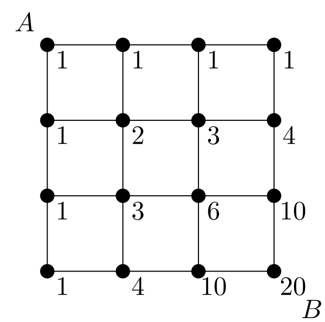
8. Noah walks from point A to point B on the grid below.



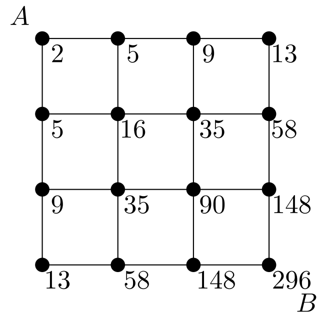
He tries to take the shortest path (that is, walk only rightwards and downwards). Unfortunately, Noah has an imperfect sense of direction, so he walks *exactly once* in the wrong direction (that is, either leftwards or upwards). He always stays in the depicted grid. How many possible paths could Noah have taken? (Note that he only reaches point B once).

8. 296

Solution: We first compute the solution assuming that Noah cannot walk back at any point. We compute this by writing the number of ways to get to each point in the grid, where the number of ways to reach any point is the sum of the number of ways to reach the point directly above and directly to the left. Denote this grid by G_0.



We then perform the same process on a new grid, call this grid G_1, where the value at each node is given by the sum of the values at the node directly above and node directly to the left, plus the values **in** G_0 at the node directly below and directly to the right. You can think of G_1 as a grid above G_0, where the only edges between G_0 and G_1 correspond to the possible backward movements. Then, it suffices to compute the number of ways to travel from point A in G_0 and point B in either G_0 or G_1. Note that we must take special care to not account for the paths which reach B, backtrack one step, and return to B. We compute G_1 as follows.



Thus, we conclude that there are 296 paths to B.

9. The value of $2\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{7}\right)$ can be expressed in the form $\frac{a}{b}\pi$, where a and b are relatively prime positive integers. Find $a+b$.

9. 5

Solution: Recall that the angle between the complex number $x+iy$ and the x-axis is the arctangent of $\frac{y}{x}$. Furthermore, if we have another complex number $u+iv$, then the angle between $(x+iy)(u+iv)$ and the x-axis is

$$\arctan\left(\frac{\operatorname{Re}\left((x+iy)(u+iv)\right)}{\operatorname{Im}\left((x+iy)(u+iv)\right)}\right).$$

So, because

$$\begin{aligned}(3+i)(3+i)(7+i) &= (8+6i)(7+i)\\ &= 50+50i,\end{aligned}$$

it follows that

$$\begin{aligned}2\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{7}\right) &= \arctan\left(\frac{\operatorname{Re}(50+50i)}{\operatorname{Im}(50+50i)}\right)\\ &= \arctan(1)\\ &= \frac{1}{4}\pi.\end{aligned}$$

So, our solution is $1+4=5$.

10. Triangle $\triangle ABC$ has dimensions $BC = a$, $CA = b$, $AB = c$. If a, b, c forms a geometric progression, and $\sin(\angle B - \angle A), \sin(\angle A), \sin(\angle C)$ forms a arithmetic progression, determine $\cos(\angle B)$.

10. $\frac{\sqrt{5}-1}{2}$

Solution: Since a, b, and c form a geometric progression, we can write $b = ra$ and $c = r^2a$. By the law of cosines,

$$\begin{aligned}\cos(A) &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{r^2 + r^4 - 1}{2r^3}; \\ \cos(B) &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{1 + r^4 - r^2}{2r^2}.\end{aligned}$$

By the law of sines, we have

$$\begin{aligned}\sin(B) &= \frac{b \sin(A)}{a} \\ &= r \sin(A); \\ \sin(C) &= \frac{c \sin(A)}{a} \\ &= r^2 \sin(A).\end{aligned}$$

Since $\sin(B - A)$, $\sin(A)$, and $\sin(C)$ form an arithmetic progression, we can see

$$\begin{aligned}2\sin(A) &= \sin(B - A) + \sin(C) \\ &= \sin(B)\cos(A) - \sin(A)\cos(B) + \sin(C) \\ &= r \sin(A) \cdot \frac{r^2 + r^4 - 1}{2r^3} - \sin(A) \cdot \frac{1 + r^4 - r^2}{2r^2} + r^2 \sin(A) \\ 4r^2 &= 2r^4 + 2r^2 - 2 \\ 0 &= r^4 - r^2 - 1 \\ r^2 &= \frac{1 + \sqrt{5}}{2}.\end{aligned}$$

Note that we take the positive solution to the quadratic since $r^2 \geq 0$. Continuing,

$$\begin{aligned}\cos(B) &= \frac{1 + r^4 - r^2}{2r^2} \\ &= \frac{2}{2r^2} \\ &= \frac{2}{1 + \sqrt{5}} \\ &= \frac{\sqrt{5} - 1}{2}.\end{aligned}$$