Proposition 1: Power Round

Names: ____________________

Team ID: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve the problems during this round.
3. Your submission will be graded and assigned point values out of the total points possible per problem. Your total score will be the sum of the points you receive for each problem.
4. Submissions will be graded on correctness as well as clarity of proof. A proof with significant progress towards a solution may receive more credit than a correct answer with no justification.
5. **You may use the result of a previous problem in the proof of a later problem, even if you do not submit a correct solution to the referenced problem.** However, you may not use the result of a later problem in the proof of an earlier problem.
6. Please submit each part of each problem on a separate page. Write your team ID, problem number, and page number clearly at the top of each page.
7. No calculators or electronic devices are allowed.
8. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
9. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
10. Do not discuss the problems with anyone outside of your team until all papers have been collected.
11. If you have a question or need to leave the room for any reason, please raise your hand quietly.
12. Good luck!

Acceptable Answers

1. Solutions should be written in proof format. All answers, reasoning, and deductions must be explained and justified, unless the problem explicitly asks for you to "compute". Problems asking you to "show", "prove", or "justify" **require proof!**

2. Proofs will be graded both on correctness as well as clarity of presentation.

3. Partial credit may be awarded for significant progress towards a solution.

4. Each problem must be written starting on a new, blank page. Two different problems should not be written on the same page.

5. At the top right corner of each page, please clearly print your Team ID, problem number, and page number. **Do not write your Team Name.**

6. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

1 Isoperimetric Problems

Problem 1.1 (2 points)**.** Suppose we have a rod of length L. We want to cut up the rod into four pieces (two of length x, two of length y), and then assemble those four pieces to form a rectangle. What values of x and y will maximize the area of the rectangle?

These types of problems are often called isoperimetric problems: among all shapes (of some type) with the same perimeter, find the one with the maximum area. The most famous one is Dido's problem: show that among all "shapes" (regions in the plane) with the same perimeter, a circle has the maximum area. This problem is a tricky one: your region could be very very complicated. Even solving Dido's problem in the case where the region is "nice" (smooth) required some complicated calculus. However, using the below logic, one can often get reasonably far with problems of this type.

Problem 1.2 (2 points)**.** Show that (in the context of the previous problem) making the shorter side shorter and the longer side longer decreases the area.

Problem 1.3 (2 points)**.** Suppose we instead wish to assemble a rectangular prism: cutting the rod into four pieces of length x, four of length y, and four of length z. What values of x, y, and z will maximize the volume the rectangular prism? What is the maximum volume?

2 Equal Sum-Product Problem

We now turn our attention to an interesting problem, which at first glance seems unrelated to the isoperimetric problem. It is well known that $2 + 2 = 2 \times 2$, and similarly, $1 + 2 + 3 = 1 \times 2 \times 3$. This motivates us to ask the following question:

Q: Let n be a positive integer. How many ways can we find $(a_1, a_2 \ldots, a_n)$, where each a_i is a positive integer, and $a_1 \leq a_2 \leq \cdots \leq a_n$, such that the condition

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$$

is satisfied?

(Note that the a_i's are **positive integers**.) For example, from above we know that when $n = 2$, the collection $(2, 2)$ is a solution, and for $n = 3$, the collection $(1, 2, 3)$ is a solution. Numbers other than 1 make the sum bigger, but make the product much much bigger. However, multiplying by 1 will not change the product, while adding 1 will increase the sum: 1s pull in the opposite direction, making the sum larger while leaving the product unchanged.

Problem 2.1 (6 points)**.** Show that given an integer N, there is some $n > 1$ such that we can find $a_1, \ldots, a_n$ with $N = \sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$ if and only if N is composite.

Problem 2.2 (4 points)**.** For any integer $n \geq 1$, show that at least one solution exists, by constructing an explicit solution.

Your solution above will likely involve lots of 1s, which motivates us to consider the number of ones in an arbitrary solution.

Problem 2.3 (8 points)**.** Show that for $n \geq 3$, any solution must contain at least one 1.

We now know enough to solve the problem completely for $n = 3$.

Problem 2.4 (4 points)**.** Show that $(1, 2, 3)$ is the only solution for $n = 3$.

We consider a few special cases.

Problem 2.5 (6 points)**.** Prove that a solution of length n exists with

- $a_i = 1$ for all $1 \leq i \leq n-2$
- $a_{n-1}, a_n > 2$

if and only if $n-1$ is composite.

Problem 2.6 (6 points)**.** Prove that if $n \equiv 2 \pmod 6$ and $n > 2$, there exists a solution with exactly three non-1 values.

Problem 2.7 (6 points)**.** Suppose n is even. Show that any solution $S = \sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$ necessarily satisfies $S \equiv 0 \pmod 4$.

We now attempt to tackle the general case.

Problem 2.8 (10 points)**.** Prove that if $n > 1$, $(a_1, a_2, \dots, a_n)$ is a solution, and

$$S = \sum_{i=1}^{n} a_i,$$

then $S \leq 2n$. *Hint*: Remember Problem 1.2.

Problem 2.9 (4 points)**.** Show that if $(a_1, \dots, a_n)$ is a solution, then $a_i \leq n+1$ for all i.

Problem 2.10 (2 points)**.** Show further that if $n > 1$, then $a_i \leq n$ for all i.

We revisit the idea that we would like our solutions to contain many occurrences of 1.

Problem 2.11 (4 points)**.** Suppose that a solution contains exactly k occurrences of 1, that is,

$$a_1 = a_2 = \cdots = a_k = 1,$$
$$a_i \neq 1 \text{ for all } k+1 \leq i \leq n.$$

Show that $k \geq n - \log_2(n) - 1$.

Using both Problem 2.8 and Problem 2.11, we can reduce the number of possible solutions to a number which is reasonably searchable by hand.

Problem 2.12 (4 points)**.** Compute all solutions for each of $n = 4$, $n = 5$, $n = 6$, and $n = 7$.

A follow-up question may be to ask how the number of solutions behaves as n grows.

Problem 2.13 (10 points)**.** Prove that for any integer M, we can find n such that the number of n-tuple solutions is at least M.

3 A Problem with Tangents

Problem 3.1 (4 points)**.** Prove that for any triangle $\triangle ABC$, we have

$$\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C).$$

Problem 3.2 (2 points)**.** How many triangles, up to rescaling, have angles that all have integer tangents? Justify your answer.