Proposition 1: Power Round

Names: ______________________________

Team ID: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve the problems during this round.
3. Your submission will be graded and assigned point values out of the total points possible per problem. Your total score will be the sum of the points you receive for each problem.
4. Submissions will be graded on correctness as well as clarity of proof. A proof with significant progress towards a solution may receive more credit than a correct answer with no justification.
5. **You may use the result of a previous problem in the proof of a later problem, even if you do not submit a correct solution to the referenced problem.** However, you may not use the result of a later problem in the proof of an earlier problem.
6. Please submit each part of each problem on a separate page. Write your team ID, problem number, and page number clearly at the top of each page.
7. No calculators or electronic devices are allowed.
8. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
9. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
10. Do not discuss the problems with anyone outside of your team until all papers have been collected.
11. If you have a question or need to leave the room for any reason, please raise your hand quietly.
12. Good luck!

Acceptable Answers

1. Solutions should be written in proof format. All answers, reasoning, and deductions must be explained and justified, unless the problem explicitly asks for you to "compute". Problems asking you to "show", "prove", or "justify" **require proof!**
2. Proofs will be graded both on correctness as well as clarity of presentation.
3. Partial credit may be awarded for significant progress towards a solution.
4. Each problem must be written starting on a new, blank page. Two different problems should not be written on the same page.
5. At the top right corner of each page, please clearly print your Team ID, problem number, and page number. **Do not write your Team Name.**
6. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

1 Isoperimetric Problems

Problem 1.1 (2 points)**.** Suppose we have a rod of length L. We want to cut up the rod into four pieces (two of length x, two of length y), and then assemble those four pieces to form a rectangle. What values of x and y will maximize the area of the rectangle?

Solution: We want to maximize xy subject to $2x + 2y = L$. From the equation, we have

$$xy = x\left(\frac{L}{2} - x\right).$$

This quadratic has a negative leading coefficient (namely -1), and it has zeros at 0 and $\frac{L}{2}$. It follows that it is maximized when $x = \frac{1}{2}\left(0 + \frac{L}{2}\right) = \frac{L}{4}$. From our original constraint, we also require $y = \frac{L}{4}$. Note that we maximize the area precisely when $x = y$.

These types of problems are often called isoperimetric problems: among all shapes (of some type) with the same perimeter, find the one with the maximum area. The most famous one is Dido's problem: show that among all "shapes" (regions in the plane) with the same perimeter, a circle has the maximum area. This problem is a tricky one: your region could be very very complicated. Even solving Dido's problem in the case where the region is "nice" (smooth) required some complicated calculus. However, using the below logic, one can often get reasonably far with problems of this type.

Problem 1.2 (2 points)**.** Show that (in the context of the previous problem) making the shorter side shorter and the longer side longer decreases the area.

Solution: Without loss of generality, say x is the shorter side and y is the longer side. If we change both by an amount $\varepsilon > 0$, then the area is

$$\begin{aligned}(x-\varepsilon)(y+\varepsilon) &= xy - (y-x)\varepsilon - \varepsilon^2 \\ &< xy - (y-x)\varepsilon \\ &\leq xy.\end{aligned}$$

The last line comes from the fact that $y - x \geq 0$.
So, we see that increasing the longer side by ε and decreasing the shorter side by ε must decrease the total area of the rectangle.

Problem 1.3 (2 points)**.** Suppose we instead wish to assemble a rectangular prism: cutting the rod into four pieces of length x, four of length y, and four of length z. What values of x, y, and z will maximize the volume the rectangular prism? What is the maximum volume?

Solution: Note that we must have $2x + 2y = \frac{L}{2} - 2z$ and we want to maximize xyz. If we imagine z as a fixed variable, then we really have $2x + 2y = L_z = \frac{L}{2} - 2z$ and we want to maximize xy. The idea here is we first maximize over x and y, and then we maximize over z. From Problem 1.1, we see that xy is maximized when $x = y$. Summarizing, when xyz is maximized, we must have $x = y$.
We can similarly imagine y as a fixed variable. Repeating the above argument, we see that when xyz is maximized, we must have $x = z$. Combining, xyz is maximized only if $x = y = z$, which means the dimensions that maximize the volume are $x = y = z = \frac{L}{12}$. The volume is then $\frac{L^3}{1728}$.

2 Equal Sum-Product Problem

We now turn our attention to an interesting problem, which at first glance seems unrelated to the isoperimetric problem. It is well known that $2+2=2\times 2$, and similarly, $1+2+3=1\times 2\times 3$. This motivates us to ask the following question:

Q: Let n be a positive integer. How many ways can we find $(a_1, a_2 \ldots, a_n)$, where each a_i is a positive integer, and $a_1 \leq a_2 \leq \cdots \leq a_n$, such that the condition

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$$

is satisfied?

(Note that the a_i's are **positive integers**.) For example, from above we know that when $n=2$, the collection $(2,2)$ is a solution, and for $n=3$, the collection $(1,2,3)$ is a solution. Numbers other than 1 make the sum bigger, but make the product much much bigger. However, multiplying by 1 will not change the product, while adding 1 will increase the sum: 1s pull in the opposite direction, making the sum larger while leaving the product unchanged.

Problem 2.1 (6 points)**.** Show that given an integer N, there is some $n>1$ such that we can find $a_1, \ldots, a_n$ with $N=\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$ if and only if N is composite.

Solution: Suppose N is composite. Write $N=xy$, where $2 \leq x \leq y$. Note that $x+y \leq 2y \leq xy$. So, we can let $n = xy - x - y + 2$. From our earlier work, we have $n>1$. Define $a_{n-1}=x$ and $a_n = y$. Furthermore, let $a_i = 1$ for $1 \leq i \leq n-2$. Then

$$\begin{aligned}\sum_{i=1}^{n} a_i &= (n-2)+x+y \\ &= (xy-x-y+2-2)+x+y \\ &= xy \\ &= N,\end{aligned}$$

and

$$\begin{aligned}\prod_{i=1}^{n} a_i &= 1\cdots 1\cdot x\cdot y \\ &= xy \\ &= N,\end{aligned}$$

as desired.
Now, consider the case when N is not composite. Note that for $N=\prod_{i=1}^{n} a_i$, we must have that $a_i = N$ for one of the i's. But then $\sum_{i=1}^{n} a_i > N$, since each $a_i \geq 1$ and $n>1$. So no solution exists in this case.

Problem 2.2 (4 points)**.** For any integer $n \geq 1$, show that at least one solution exists, by constructing an explicit solution.

Solution: First, note that if $n = 1$, any choice of a_1 works as a solution.
Now, suppose $n \geq 2$. Define $a_{n-1} = 2$ and $a_n = n$. Further define $a_i = 1$ for $1 \leq i \leq n-2$ (note that this last definition is "vacuous" if $n = 2$). Then

$$\begin{aligned}\sum_{i=1}^{n} a_i &= (n-2) + 2 + n \\ &= 2n,\end{aligned}$$

and

$$\begin{aligned}\prod_{i=1}^{n} a_i &= 1 \cdots 1 \cdot 2 \cdot n \\ &= 2n,\end{aligned}$$

as desired.

Your solution above will likely involve lots of 1s, which motivates us to consider the number of ones in an arbitrary solution.

Problem 2.3 (8 points)**.** Show that for $n \geq 3$, any solution must contain at least one 1.

Solution: We will show that, if $a_i \geq 2$ for each $1 \leq i \leq n$, then

$$\sum_{i=1}^{n} a_i < \prod_{i=1}^{n} a_i.$$

Our strategy is to use induction in two different ways: one to increase n and one to increase a single a_i. So as not to get lost in notation, we will treat some steps mildly informally.
First, our base case is when $n = 3$ and $a_1 = a_2 = a_3 = 2$. Here, we just find that the sum is 6 and the product is 8, which is good since $6 < 8$.
Next, if we increase n by 1 by adding in $a_{n+1} = 2$, we see immediately that the sum increases by $a_{n+1} = 2$. Since $\prod_{i=1}^{n} a_i \geq a_1 \geq 2$, the product increases by at least 2. So, if

$$\sum_{i=1}^{n} a_i < \prod_{i=1}^{n} a_i,$$

then

$$\sum_{i=1}^{n+1} a_i < \prod_{i=1}^{n+1} a_i,$$

as desired.
Now, consider increasing one of the a_i's by 1. Say we replace a_j with $a_j + 1$. In this case, the sum increases by 1. The product increases by

$$\prod_{\substack{i=1 \\ i \neq j}}^{n} a_i > 1.$$

So, here as well the result holds.

Finally, we can construct any solution to

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$$

with $a_i \geq 2$ for each $1 \leq i \leq n$ by starting with our base case, adding in as many a_i terms as we need, and then incrementing the a_i's appropriately, starting with a_n and moving leftwards. It would then follow from the above reasoning that

$$\sum_{i=1}^{n} a_i < \prod_{i=1}^{n} a_i,$$

meaning that $(a_1, \ldots, a_n)$ was not actually a solution in the first place.
As noted at the beginning of this proof, we have not been completely formal. In particular, the final step requires ugly reasoning to be completely rigorous, which is omitted.

We now know enough to solve the problem completely for $n = 3$.

Problem 2.4 (4 points)**.** Show that $(1, 2, 3)$ is the only solution for $n = 3$.

Solution: From Problem 2.3, we know that any solution for $n = 3$ must be of the form $(1, x, y)$ for some $x \leq y$. The condition for our solution is exactly $1 + x + y = xy$. Solving for x, we see that $x = \frac{y+1}{y-1}$. x must be an integer, so $y + 1$ must be divisible by $y - 1$. Consequently, $y - 1 \leq 2$. We now just check the cases $y = 2$ and $y = 3$ ($y = 1$ was already ruled out since $1 < x \leq y$). The former yields $x = 3 > y$, and the latter yields the desired solution.

We consider a few special cases.

Problem 2.5 (6 points)**.** Prove that a solution of length n exists with

- $a_i = 1$ for all $1 \leq i \leq n - 2$
- $a_{n-1}, a_n > 2$

if and only if $n - 1$ is composite.

Solution: Suppose we have a solution $(1, \ldots, 1, x, y)$ of length n such that $x, y \neq 2$. Our sum is then $n - 2 + x + y$, and our product is then xy. Equating these two and solving for x, we see $x = 1 + \frac{n-1}{y-1}$. Since x is an integer, we must have that $y - 1$ divides $n - 1$. By our assumption, we have $y \neq 2$, so $y - 1 \neq 1$. It follows that $y - 1$ must be a factor of $n - 1$. If $y - 1 = n - 1$, then we can compute $x = 2$, which is contrary to our assumption. So we furthermore know that $y - 1$ is a *non-trivial* factor of $n - 1$. It follows that $n - 1$ is composite.
Now we have to show the other direction. Suppose $n - 1 = uv$, where $1 < u \leq v < n - 1$ (that is, $n - 1$ is composite). Consider the n-tuple $(1, \ldots, 1, u + 1, v + 1)$. This has a sum of $n + u + v$. Its product is

$$\begin{aligned} (u + 1)(v + 1) &= uv + u + v + 1 \\ &= (n - 1) + u + v + 1 \\ &= n + u + v. \end{aligned}$$

It follows that this n-tuple is a solution, as desired.

Problem 2.6 (6 points)**.** Prove that if $n \equiv 2 \pmod 6$ and $n > 2$, there exists a solution with exactly three non-1 values.

Solution: We construct a solution directly. Suppose $n = 6k+2$. Our solution is $(1, \ldots, 1, 2, 2, 2k+1)$. Here, we have exactly $6k-1$ elements that are 1. Our sum is then $8k+4$. The product is also $2 \cdot 2 \cdot (2k+1) = 8k+4$.

Problem 2.7 (6 points)**.** Suppose n is even. Show that any solution $S = \sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$ necessarily satisfies $S \equiv 0 \pmod 4$.

Solution: Suppose all of the a_i's were odd. If this were the case, then the sum would be even (as n is even), but the product would be odd. So, at least one of the a_i's is even.
Now suppose exactly one of the a_i's is even. If this were the case, then the sum would be odd (as $n-1$ is odd), but the product would be even. So at least two of the a_i's is even.
Now, we note that S is the product of all of the a_i's. Since at least two of them are even, S must be doubly even.

We now attempt to tackle the general case.

Problem 2.8 (10 points)**.** Prove that if $n > 1$, $(a_1, a_2, \ldots, a_n)$ is a solution, and

$$S = \sum_{i=1}^{n} a_i,$$

then $S \leq 2n$. *Hint*: Remember Problem 1.2.

Solution: Suppose $(a_1, \ldots, a_n)$ is a solution. Our idea is to repeatedly modify this tuple. Throughout the modifications, the sum will remain the same, but the product will decrease.
First, we will show that $a_{n-1} \geq 2$. Suppose not. Then $a_1 = \cdots = a_{n-1} = 1$. So our sum is $(n-1) + a_n$ and our product is a_n. Since $n > 1$, it follows that our sum and product could not be the same, so we must have actually had $a_{n-1} \geq 2$.
Now, we will describe the modification process. Let j be the smallest number for which $a_j > 1$. If $j = n-1$, then stop here. Otherwise, replace a_j with 1, and replace a_n with $a_n + a_j - 1$. Note that the sum of the a_i's remains the same and the sequence of a_i's remains non-decreasing. By Problem 1.2, the product of the a_i's must decrease. We now repeat this process until our tuple of a_i's looks like $(1, \ldots, 1, a_{n-1}, a_n)$.
Our next step is to replace a_{n-1} with 2 and a_n with $a_n + a_{n-1} - 2$. This again keeps the sum of the a_i's the same while not increasing the product (since $a_{n-1} \geq 2$). We now have an n-tuple that looks like $(1, \ldots, 1, 2, S-n)$. By our reasoning, it's product is at most S. But it's product is $2S - 2n$. It follows that $S \leq 2n$.

Problem 2.9 (4 points)**.** Show that if $(a_1, \dots, a_n)$ is a solution, then $a_i \le n+1$ for all i.

Solution: Applying the previous problem,

$$\begin{aligned} 2n &\ge a_1 + \cdots + a_{n-1} + a_n \\ &\ge \overbrace{1 + \cdots + 1}^{n-1 \text{ times}} + a_n \\ &= n - 1 + a_n \\ n+1 &\ge a_n. \end{aligned}$$

Since a_n is the greatest of the a_i's, the result follows.

Problem 2.10 (2 points)**.** Show further that if $n > 1$, then $a_i \le n$ for all i.

Solution: Due to the previous problem, we only have to consider the case that $a_n = n+1$. Because we know that $2n \ge a_1 + \cdots + a_{n-1} + a_n$, we have $n - 1 \ge a_1 + \cdots + a_{n-1}$. Recalling that we stipulate that each $a_i \ge 1$, it follows that $a_1 = \cdots = a_{n-1} = 1$. But then the product of our tuple is $n+1$, with the sum being $2n$. This would force $n = 1$.

We revisit the idea that we would like our solutions to contain many occurrences of 1.

Problem 2.11 (4 points)**.** Suppose that a solution contains exactly k occurrences of 1, that is,

$$a_1 = a_2 = \cdots = a_k = 1,$$

$$a_i \ne 1 \text{ for all } k+1 \le i \le n.$$

Show that $k \ge n - \log_2(n) - 1$.

Solution: If we have exactly k occurrences of 1, then we have $n-k$ occurrences of numbers that are greater than 1. It follows that the product of our tuple is at least 2^{n-k}. Recalling Problem 2.8, we have

$$\begin{aligned} 2n &\ge 2^{n-k} \\ 1 + \log_2(n) &\ge n - k \\ k &\ge n - \log_2(n) - 1. \end{aligned}$$

Using both Problem 2.8 and Problem 2.11, we can reduce the number of possible solutions to a number which is reasonably searchable by hand.

Problem 2.12 (4 points)**.** Compute all solutions for each of $n = 4$, $n = 5$, $n = 6$, and $n = 7$.

Solution: $n = 4$: We have from Problem 2.8 that our sum is at most $2 \times 4 = 8$. Our only solution turns out to be $(1, 1, 2, 4)$.
$n = 5$: We have from Problem 2.8 that our sum is at most $2 \times 5 = 10$. Our possible solutions are as follows:

- $(1, 1, 1, 2, 5)$
- $(1, 1, 1, 3, 3)$
- $(1, 1, 2, 2, 2)$

$n = 6$: We have from Problem 2.8 that our sum is at most $2 \times 6 = 12$. From Problem 2.11, any tuple that is a solution has at least $6 - \log_2(6) - 1 > 2$ elements that is a 1. Our only solution turns out to be $(1, 1, 1, 1, 1, 2, 6)$.
$n = 7$: We have from Problem 2.8 that our sum is at most $2 \times 7 = 14$. From Problem 2.11, any tuple that is a solution has at least $7 - \log_2(7) - 1 > 3$ elements that is a 1. Our solutions are as follows:

- $(1, 1, 1, 1, 1, 2, 7)$
- $(1, 1, 1, 1, 1, 3, 4)$

A follow-up question may be to ask how the number of solutions behaves as n grows.

Problem 2.13 (10 points)**.** Prove that for any integer M, we can find n such that the number of n-tuple solutions is at least M.

Solution: Consider $n = 2^{2M-2} + 1$. For each $1 \leq k \leq M$, consider the n-tuple

$$(\overbrace{1, \ldots, 1}^{n-2 \text{ times}}, 2^{k-1} + 1, 2^{2M-k-1} + 1).$$

Then the sum is $n + 2^{k-1} + 2^{2M-k-1}$, and the product is

$$\begin{aligned}\left(2^{k-1} + 1\right)\left(2^{2M-k-1} + 1\right) &= 2^{2M-2} + 2^{k-1} + 2^{2M-k-1} + 1 \\ &= n + 2^{k-1} + 2^{2M-k-1}.\end{aligned}$$

Since there are M options for k and each tuple had the same length, it follows that that there are at least M solutions of length n.

3 A Problem with Tangents

Problem 3.1 (4 points)**.** Prove that for any triangle $\triangle ABC$, we have

$$\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C).$$

Solution: Recalling that $\tan(\pi - x) = -\tan(x)$ and $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$, and using the fact that $A + B + C = \pi$, we have

$$\begin{aligned}\tan(C) &= \tan(\pi - A - B) \\ &= -\tan(A + B) \\ &= -\frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}.\end{aligned}$$

Now, we can compute

$$\begin{aligned}
\tan(A)+\tan(B)+\tan(C) &= \tan(A)+\tan(B)-\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}\\
&= (\tan(A)+\tan(B))\left(1-\frac{1}{1-\tan(A)\tan(B)}\right)\\
&= -\left(\tan(A)+\tan(B)\right)\frac{\tan(A)\tan(B)}{1-\tan(A)\tan(B)}\\
&= \tan(A)\tan(B)\left(-\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}\right)\\
&= \tan(A)\tan(B)\tan(C).
\end{aligned}$$

Problem 3.2 (2 points)**.** How many triangles, up to rescaling, have angles that all have integer tangents? Justify your answer.

Solution: From the previous problem and Problem 2.4, we know that the only possibility for $\triangle ABC$ (up to relabeling) is when $\tan(A)=1$, $\tan(B)=2$, and $\tan(C)=3$. So there is only 1 such triangle, up to rescaling.

References

[1] M. W. Ecker. *When Does a Sum of Positive Integers Equal Their Product?*, Mathematics Magazine Vol. 75, No. 1, February 2002, `https://www.researchgate.net/publication/270185488_When_Does_a_Sum_of_Positive_Integers_Equal_Their_Product`

[2] L. Kurlandchik, A. Nowicki. *When the sum equals the product*, `https://www-users.mat.umk.pl/~anow/ps-dvi/si-krl-a.pdf`

[3] M. A. Nyblom, C. D. Evans. *An Algorithm to Solve the Equal-Sum-Product Problem*, `https://arxiv.org/abs/1311.3874`