Proposition 5: Tiebreaker Round

Name: ____________________

Team ID: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 30 minutes to solve 3 problems.
3. When you would like to submit your answers, please inform your proctor.
4. **Your score will be the number of correct answers, with ties broken by time of submission.**
5. No calculators or electronic devices are allowed.
6. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Proposition 5: Tiebreaker Round

1. Luke writes a rational number on each of 4 cards, denoted a_1, a_2, a_3, a_4. Luke computes all possible products of two different cards, and tells you the six products he gets are

$$\left\{-45, -\frac{9}{5}, -\frac{5}{3}, -\frac{1}{15}, 3, 1\right\}.$$

Determine all possible values of $a_1 + a_2 + a_3 + a_4$.

1. $-\frac{62}{15}, \frac{62}{15}$

Solution: We first start by trying to find what $a_1a_2a_3a_4$ is. Note that the product of the 6 given numbers is $(a_1a_2a_3a_4)^3 = 27$, so $a_1a_2a_3a_4 = 3$.

Without loss of generality, we say $a_1a_2 = -45$. Then we must have $a_3a_4 = -\frac{1}{15}$. Still without losing generality, we can also say $a_1a_3 = 3$ and $a_2a_4 = 1$. We then have two cases to consider:

- Suppose $a_1a_4 = -\frac{5}{3}$. Then

$$\begin{aligned} -\frac{5}{3} &= a_1a_4 \\ &= \frac{a_1}{a_2} \\ &= -\frac{a_1^2}{45} \\ a_1^2 &= 75. \end{aligned}$$

  Since a_1 is rational, this case cannot happen.

- Suppose $a_1a_4 = -\frac{9}{5}$. Then

$$\begin{aligned} -\frac{9}{5} &= a_1a_4 \\ &= \frac{a_1}{a_2} \\ &= -\frac{a_1^2}{45} \\ a_1^2 &= 81 \\ a_1 &= \pm 9. \end{aligned}$$

  This then yields $a_2 = \mp 5$, $a_3 = \pm\frac{1}{3}$, and $a_4 = \mp\frac{1}{5}$. We can verify that both of these solutions yield the 6 products required.

It follows that our answers are

$$\pm\left(9 - 5 + \frac{1}{3} - \frac{1}{5}\right) = \pm\frac{62}{15}.$$

2. Consider the grid below.

How many ways can we place the integers $\{1, 2, \dots, 9\}$, without replacement, into the grid, such that the sum of each row and the sum of each column is divisible by 3?

2. 5184

Solution: We consider solutions modulo 3. This amounts to placing three 0's, three 1's, and three 2's into the grid such that each row and each column sums to 0 modulo 3. There are two ways to do this:

1. Each row (alternatively, column) contains either all 0's, all 1's, or all 2's.
2. Each row and each column contains exactly one of each number.

In Case 1, we can choose either rows or columns to work with, so without loss of generality, consider rows. We can choose $3! = 6$ ways to order a row of all 0's, a row of all 1's, and a row of all 2's. Since we can choose either rows or columns, we have a total of 12 cases.

In Case 2, we can choose any permutation of $\{0, 1, 2\}$ for the first column, so without loss of generality, choose $(0, 1, 2)$. Note that there are $3! = 6$ choices here.

0	$*$	
1		
2		

Note that in the position of $*$, we cannot choose 0, so only 1 and 2 are possible. This then uniquely determines each board as follows.

0	1	2
1	2	0
2	0	1

0	2	1
1	0	0
2	1	2

Therefore, for each choice of the first column, there are exactly 2 possible boards, so the total number of boards in this case is $3! \cdot 2 = 12$.

Therefore, there are exactly $12 + 12 = 24$ configurations of three 0's, three 1's, and three 2's such that each row and column adds to 0 modulo 3.

We then replace all instances of 0 with an element of $\{3, 6, 9\}$, all instances of 1 with an element of $\{1, 4, 7\}$, and all instances of 2 with an element of $\{2, 5, 8\}$. There are $3! \cdot 3! \cdot 3! = 6^3$ ways to do this. Thus, our final answer is given by $6^3(24) = 5184$.

3. Let S_1 be a sphere of radius 1. Inscribe in S_1 a regular tetrahedron T_1. Inside T_1, inscribe a sphere S_2. Inside S_2, inscribe a regular tetrahedron T_2. Inside T_2, inscribe a sphere S_3. Inside S_3, inscribe a regular tetrahedron T_3. What is the volume of T_3?

3. $\frac{8\sqrt{3}}{19683}$

Solution: Note that the vertices of a regular tetrahedron form a subset of the vertices of a cube. In particular, if a tetrahedron has the vertices

$$\{(0,0,0),(0,s,s),(s,0,s),(s,s,0)\},$$

then the cube has vertices

$$\{(0,0,0),(0,0,s),(0,s,0),(0,s,s),(s,0,0),(s,0,s),(s,s,0),(s,s,s)\}.$$

The center of the circumscribing sphere for the tetrahedron is then just the center of the cube, which is $\left(\frac{s}{2},\frac{s}{2},\frac{s}{2}\right)$. So the circumscribing sphere has a radius $R=\frac{s\sqrt{3}}{2}$.

The center of the inscribed sphere for the tetrahedron is also $\left(\frac{s}{2},\frac{s}{2},\frac{s}{2}\right)$. Let A be the area of one face of the regular tetrahedron, and let r be the radius of the inscribed sphere. Note that $R+r$ is the height of the regular tetrahedron. Also, the regular tetrahedron is composed of four copies of the tetrahedron with vertices

$$\left\{(0,0,0),(0,s,s),(s,0,s),\left(\frac{s}{2},\frac{s}{2},\frac{s}{2}\right)\right\},$$

so we can compare volumes to see that

$$\begin{aligned}\frac{1}{3}A(R+r) &= 4\cdot\frac{1}{3}Ar\\ R &= 3r.\end{aligned}$$

It follows that the radius of S_3 is one third of that of S_2, which is one third of that of S_1. Thus, the radius of S_3 is $\frac{1}{9}$.

Now, it remains to find the volume V of T_3. Let's mildly change the above notation. We now have $R=\frac{1}{9}$, so $s=\frac{2}{9\sqrt{3}}$. T_3 is just our cube with four copies of the tetrahedron with vertices

$$\{(0,0,0),(0,s,s),(s,0,s),(0,0,s)\}$$

cut out. Comparing the volumes, we have

$$\begin{aligned}s^3 &= V+4\cdot\frac{1}{3}\left(\frac{1}{2}s^2\right)\cdot s\\ &= V+\frac{2}{3}s^3\\ V &= \frac{1}{3}s^3\\ &= \frac{1}{3}\left(\frac{2}{9\sqrt{3}}\right)^3\\ &= \frac{8\sqrt{3}}{19683}.\end{aligned}$$