Proposition 4: Wolverine Round

Names: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 30 problems, given in sets of 3. You will receive the next set of problems when you turn in your answers to the previous set.
3. Your score will be the total point value of the problems you answer correctly. There is no penalty for guessing or incorrect answers.
4. Later problems will generally be more difficult, and thus worth more points. **It may be to your advantage to skip earlier problems in order to conserve time!**
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.

2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.

3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.

4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.

5. Answers should be expressed in base 10 unless otherwise specified.

6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.

7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.

8. Angle measurements should be given in radians unless otherwise specified.

9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

1. Determine

$$\log_2(3)\log_9(343)\log_{49}(1024)$$

1. ______________

2. Let x and y be integers such that

$$\begin{aligned} \log_3 x + \log_9 y &= 7 \\ \log_y x &= 3 \end{aligned}$$

Determine $x + y$.

2. ______________

3. In preparation for the Area 52 raid, the US needs to prepare a defense force against the raid. Suppose that one officer can deal with exactly 52 raiders. On the Bookface page, the raid has the support of 100,000 people. However, only 1 out of every 100 supporters will actually go to the raid. What is the minimum number of troops the US needs to send to Area 52 to successfully hold of the raiders?

3. ______________

4. Suppose that you are applying for a license plate. Since you want to be as annoying as possible, you want your license plate to be completely composed of the characters "B" and "8". The minimum number of characters on a license plate is 5 and the maximum is 8. How many possible license plates can you create?

4. ______________

5. Compute

$$\prod_{n=0}^{\infty} \pi^{1/2^n}$$

5. ______________

6. A triangle has integer side lengths 2, p, and q, where p and q are distinct primes. What is the greatest possible perimeter of this triangle?

6. ______________

7. Steven writes the numbers 1 to 2020 on a very large blackboard. He then erases two numbers at random and writes the sum of the numbers on the blackboard. He repeats this process until there is only 1 number left on the board. What is that number?

7. ______________

8. Triangle $\triangle ABC$ has dimensions $AB = 10$, $BC = 17$, and $AC = 21$. Let α be the angle opposite side $\overline{BC}$. Determine $\tan(\alpha)$.

8. ______________

9. There is some integer $n = \sqrt{342 + \sqrt{342 + \sqrt{342 + \sqrt{342 + \ldots}}}}$. Find n.

9. ______________

10. Let $f(x) = ax^2 + bx + c$ be a polynomial with integer coefficients and $a \neq 0$. Suppose that

$$f(2020) = f(-2020)$$

and 43 is a root of f. Determine $\frac{c}{a}$.

10. ______________

11. A triangle is formed by the points

$$a : (0, 0, 0), \qquad b : (2, 1, 0), \qquad c : (1, 2, 0),$$

in the plane. There is a unique point $d = (x, y, z)$ such that

- The point $(x, y, 0)$ is the centroid of the above triangle,
- The value $z > 0$,
- The volume of the tetrahedron formed by a, b, c, and d is equal to the area of the triangle (ignoring units).

Find z.

11. ______________

12. Six people A, B, C, D, E, F attend a staring competition, and there are six number plates with numbers $1, 2, 3, 4, 5, 6$ at the sign-in table. They each pick a number plate upon entry to determine the order they perform. If we let a denote the person A's number, b denote person B's number, etc., what is the probability that $abc + def$ is odd?

12. ______________

13. Town A and Town B are on opposite sides of a river that has parallel banks. The river runs east-to-west, and its banks are 3 kilometers apart. Town A is 5 kilometers north of the northern bank of the river, and Town B is 4 kilometers south of the southern bank. Additionally, Town B is 12 kilometers east of Town A. You build a bridge across the river perpendicular to its banks which minimizes the total length of the shortest path between the towns that crosses the bridge. What is the length of this path in kilometers?

13. ______________

14. Every morning, 8 students sit in a line and think about math. Steve, Steven, and Stephen decide to sit together so they can think about math as a team. How many ways can we order the students so that the trio are able to sit together?

14. ______________

15. Determine the number of solutions $x \in [0, 2\pi)$ satisfying

$$\sin(x)^{\cos(x)} = 1$$

15. ______________

16. Compute

$$\sum_{n=1}^{2020} \cos\left(\frac{n\pi}{2020}\right)$$

16. ______________

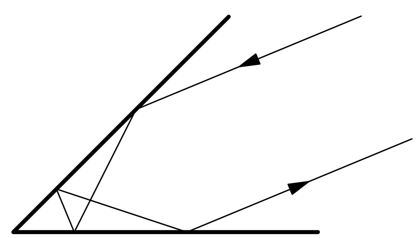
17. Nikhil likes to mine SchaeferCoin, his favorite cryptocurrency. In order to receive a SchaeferCoin, Nikhil must find a number n such that $n!$ ends in 300 zeroes. Determine the smallest possible value of n.

17. ____________

18. Stephen has 7 distinct Michigan Math T-Shirts. Every week, Stephen randomly chooses a shirt to wear on Monday, then randomly chooses a different shirt on Tuesday, etc. At the end of the week, Stephen washes all of his shirts and begins again. If there are 4 weeks in a month, determine the probability that Stephen never wears the same shirt two days in a row during the entire month.

18. ____________

19. A projectile travels in a straight line towards a wedge-shaped corner between two intersecting walls. Assume that the object is a point, and at a collision, the angle between the trajectory and the wall is the same before and after the collision. If the angle between the first wall and the incoming trajectory is equal to the angle between the second wall and the outgoing trajectory, and the incoming and outgoing trajectories are parallel, then what is the angle between the two walls?



19. ____________

20. Let p_n denote the nth prime number (i.e., $p_1 = 2$). Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that

$$f(p_n x) = p_{n+1} f(x),$$
$$f(1) = 1.$$

Determine $f\left(\frac{2019}{2020}\right)$.

20. ____________

21. Michigan's football team has the ball in the final minutes of the game! Every play is either first, second, third, or fourth down. On each play, we know that

- If it is first down, there is a $\frac{2}{5}$ chance that the next play will be first down and a $\frac{3}{5}$ chance that the next play will be second down.
- If it is second down, there is a $\frac{1}{2}$ chance that the next play will be first down and a $\frac{1}{2}$ chance that the next play will be third down.
- If it is third down, there is a $\frac{3}{5}$ chance that the next play will be first down and a $\frac{2}{5}$ chance that the next play will be fourth down.
- If it is fourth down, there is a $\frac{1}{10}$ chance the next play will be first down and a $\frac{9}{10}$ chance the game will be over!

Michigan begins on first down. Determine the probability that after 4 plays, Michigan is on first down.

21. ____________

22. Annie and Noah are playing a game with a fair coin. They flip the coin indefinitely. Annie wins if at any point, three consecutive flips are Heads, Heads, Tails. Noah wins if at any point, three consecutive flips are Tails, Heads, Tails. What is the probability Annie wins?

22. ______________

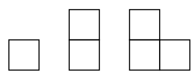
23. Matthew wants to paint each of the 6 faces of a wooden die one of red, green, or blue. How many ways are there for him to do this, where two configurations are the same if you can rotate one configuration to the other?

23. ______________

24. There exists a degree 3 polynomial f in four complex variables such that the four complex numbers z_1, z_2, z_3, z_4 form a parallelogram (when thought of as points in $\mathbb{R}^2$) if and only if $f(z_1, z_2, z_3, z_4) = 0$. Find $f(1, 6, 1, 8)/f(0, 3, 3, 9)$.

24. ______________

25. Annie's mom bought her a new jigsaw puzzle. The set contains a positioned 3×3 board (in other words, the board cannot be rotated) and an ample supply of tiles of shape:



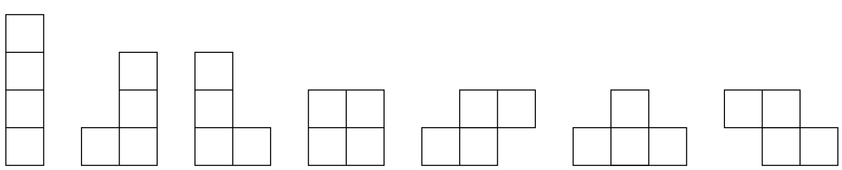
In how many different ways can she fully fill the board with the tiles if each type of tile must be used at least once?

25. ______________

26. Define a recursive sequence by $a_0 = 1$, $a_1 = 1$, and $a_n = a_{n-1}/3 + a_{n-2}/4$ for $n \geq 2$. Find $\sum_{n=0}^{\infty} a_n$.

26. ______________

27. An n-polyomino is a 2-dimensional polygon made up of n 1×1 squares, such that every square touches any other square at only edges and corners. For example, all 4-polyominos are shown below.



How many possible 5-polyominoes exist, if two polyominoes are the same if one is a rotation (but not reflection) of another?

27. ______________

28. Noah rolls three six-sided dice, numbered from 1 to 6. What is the probability that the numbers which come up can be the side lengths of a triangle?

28. ______________

29. Let x, y, z be non-negative real numbers such that $x + y + z = 1$.

$$(x + 5y + 7z)\left(x + \frac{y}{5} + \frac{z}{7}\right)$$

Let M denote the maximum of the expression above, and let N denote the minimum. Determine $M+N$.

29. ______________

30. Ani the Ant is on a vertex of a regular icosahedron with side length 2. Every second, Ani moves to one of the adjacent vertices with equal probability. After 3 seconds, what is the expected value of Ani's straight-line distance from his original location?

30. ______________

31. **(TIEBREAKER)** How many total lines of LaTeX were used in typing the Wolverine Round? Ties will be broken by the closest answer without going over.

31. ______________