Proposition 4: Wolverine Round

Names: ____________________

Team Name: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 30 problems, given in sets of 3. You will receive the next set of problems when you turn in your answers to the previous set.
3. Your score will be the total point value of the problems you answer correctly. There is no penalty for guessing or incorrect answers.
4. Later problems will generally be more difficult, and thus worth more points. **It may be to your advantage to skip earlier problems in order to conserve time!**
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.

2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.

3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.

4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.

5. Answers should be expressed in base 10 unless otherwise specified.

6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.

7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.

8. Angle measurements should be given in radians unless otherwise specified.

9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

1. Determine

$$\log_2(3)\log_9(343)\log_{49}(1024)$$

1. $\frac{15}{2}$

Solution:

$$\log_2(3)\log_9(343)\log_{49}(1024) = \frac{\log(3)\log(343)\log(1024)}{\log(2)\log(9)\log(49)} = \log_9(3)\log_{49}(343)\log_2(1024)$$
$$= \frac{1}{2}\cdot\frac{3}{2}\cdot 10 = \boxed{\frac{15}{2}}$$

2. Let x and y be integers such that

$$\log_3 x + \log_9 y = 7$$
$$\log_y x = 3$$

Determine $x + y$.

2. 738

Solution: Since $\log_y x = 3$, $x = y^3$. Then

$$\log_3 x + \log_9 y = \log_3 y^3 + \log_{\sqrt{9}} \sqrt{y} = 3\log_3 y + \frac{1}{2}\log_3 y = \frac{7}{2}\log_3 y = 7$$
$$\Rightarrow \log_3 y = 2 \Rightarrow y = 9 \Rightarrow x + y = y^3 + y = 9^3 + 9 = \boxed{738}.$$

3. In preparation for the Area 52 raid, the US needs to prepare a defense force against the raid. Suppose that one officer can deal with exactly 52 raiders. On the Bookface page, the raid has the support of 100,000 people. However, only 1 out of every 100 supporters will actually go to the raid. What is the minimum number of troops the US needs to send to Area 52 to successfully hold of the raiders?

3. 20

Solution: The number of raiders is $\frac{100{,}000}{100} = 1000$. The number of troops needed is $\lceil \frac{1000}{52} \rceil = \boxed{20}$.

4. Suppose that you are applying for a license plate. Since you want to be as annoying as possible, you want your license plate to be completely composed of the characters "B" and "8". The minimum number of characters on a license plate is 5 and the maximum is 8. How many possible license plates can you create?

4. 480

Solution: The number of n-character license plates you can create is 2^n. Thus there are $2^5 + 2^6 + 2^7 + 2^8 = 2^9 - 2^5 = 512 - 32 = \boxed{480}$ license plates.

5. Compute

$$\prod_{n=0}^{\infty} \pi^{1/2^n}$$

5. π^2

Solution: The log base π of the expression is

$$\log_\pi \left(\prod_{n=0}^{\infty} \pi^{1/2^n} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1 - 1/2} = 2.$$

Thus the original expression is $\boxed{\pi^2}$.

6. A triangle has integer side lengths 2, p, and q, where p and q are distinct primes. What is the greatest possible perimeter of this triangle?

6. 7

Solution: By the triangle inequality, we must have $|p - q| < 2$. The only pair of distinct primes for which this is the case is $(2, 3)$. Thus the triangle has side lengths $(2, 2, 3)$, and perimeter $\boxed{7}$.

7. Steven writes the numbers 1 to 2020 on a very large blackboard. He then erases two numbers at random and writes the sum of the numbers on the blackboard. He repeats this process until there is only 1 number left on the board. What is that number?

7. 2041210

Solution: At each step the sum of the numbers on the board is unchanged. Thus it suffices to find the sum of the numbers 1 to 2020.

$$\sum_{n=1}^{2020} = \frac{2020(2020+1)}{2} = 1010 \cdot 2021 = \boxed{2041210}.$$

8. Triangle $\triangle ABC$ has dimensions $AB = 10$, $BC = 17$, and $AC = 21$. Let α be the angle opposite side $\overline{BC}$. Determine $\tan(\alpha)$.

8. $\frac{4}{3}$

Solution: We drop an altitude from B to AC. Call the foot of this altitude D. Then we have two right triangles with integer side lengths: $\triangle ABD$ is a 6-8-10 triangle, and $\triangle BCD$ is an 8-15-17 triangle. Thus

$$\tan(\alpha) = \frac{BD}{AD} = \frac{8}{6} = \boxed{\frac{4}{3}}.$$

9. There is some integer $n = \sqrt{342 + \sqrt{342 + \sqrt{342 + \sqrt{342 + \ldots}}}}$. Find n.

9. 19

Solution: Since we know n exists, we can square both sides, obtaining

$$n^2 = 342 + \sqrt{342 + \sqrt{342 + \sqrt{342 + \ldots}}} = 342 + n$$
$$\Rightarrow n^2 - n - 342 = (n-19)(n+18) = 0 \Rightarrow n = \boxed{19},$$

since n is a positive integer.

10. Let $f(x) = ax^2 + bx + c$ be a polynomial with integer coefficients and $a \neq 0$. Suppose that

$$f(2020) = f(-2020)$$

and 43 is a root of f. Determine $\frac{c}{a}$.

10. -1849

Solution: Since $f(2020) = f(-2020)$, $b = 0$. We know $a \cdot 43^2 + c = 0$, so $\frac{c}{a} = -43^2 = \boxed{-1849}$.

11. A triangle is formed by the points

$$a : (0,0,0), \qquad b : (2,1,0), \qquad c : (1,2,0),$$

in the plane. There is a unique point $d = (x, y, z)$ such that

- The point $(x, y, 0)$ is the centroid of the above triangle,
- The value $z > 0$,
- The volume of the tetrahedron formed by a, b, c, and d is equal to the area of the triangle (ignoring units).

Find z.

11. 3

Solution: Let A be the area of the triangle. Then the volume of the tetrahedron can be given by

$$\frac{1}{3}(\text{base}) \cdot (\text{height}) = \frac{1}{3}A \cdot z.$$

Then $\frac{1}{3}A \cdot z = A \Rightarrow z = \boxed{3}$.

12. Six people A, B, C, D, E, F attend a staring competition, and there are six number plates with numbers $1, 2, 3, 4, 5, 6$ at the sign-in table. They each pick a number plate upon entry to determine the order they perform. If we let a denote the person A's number, b denote person B's number, etc., what is the probability that $abc + def$ is odd?

12. $\frac{1}{10}$

Solution: If at least one of (a, b, c) and at least one of (d, e, f) is even, then $abc + def$ will be even. Thus it will be odd if and only if (a, b, c) and (d, e, f) each have the same parity. There are $2 \cdot 3! \cdot 3!$ such orders, since we may choose which group has the odds in one of two ways, and we may order the three numbers within each group in one of 3! ways. The total number of orderings is 6!, so the probability $abc + def$ is odd is

$$\frac{2 \cdot 3! \cdot 3!}{6!} = \frac{2 \cdot 6 \cdot 6}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{5 \cdot 2} = \boxed{\frac{1}{10}}.$$

13. Town A and Town B are on opposite sides of a river that has parallel banks. The river runs east-to-west, and its banks are 3 kilometers apart. Town A is 5 kilometers north of the northern bank of the river, and Town B is 4 kilometers south of the southern bank. Additionally, Town B is 12 kilometers east of Town A. You build a bridge across the river perpendicular to its banks which minimizes the total length of the shortest path between the towns that crosses the bridge. What is the length of this path in kilometers?

13. 18

Solution: Since the river must be crossed perpendicularly, we may associate opposite sides of the bank. Then it becomes clear that in the resulting transformed picture, the straight line path from A to B is the shortest. By the Pythagorean theorem, the length of this path is

$$\sqrt{12^2 + (4 + 5)^2} = 15.$$

Including the length of the bridge, the total length is $15 + 3 = \boxed{18}$.

14. Every morning, 8 students sit in a line and think about math. Steve, Steven, and Stephen decide to sit together so they can think about math as a team. How many ways can we order the students so that the trio are able to sit together?

14. 4320

Solution: We first decide the order in which Steve, Steven, and Stephen sit in one of 3! ways. Then we can treat Steve, Steven, and Stephen as a unit, and order the resulting 6 units in one of 6! ways In total, there are $3! \cdot 6! = \boxed{4320}$ such orderings.

15. Determine the number of solutions $x \in [0, 2\pi)$ satisfying

$$\sin(x)^{\cos(x)} = 1$$

15. 2

Solution: For any solution x, we must have $\cos(x) = 0$ or $\sin(x) = \pm 1$.

We have $|\sin(x)| = 1$ only when $x = \pi/2, 3\pi/2$. In both these cases, $\cos(x) = 0$, so $\sin(x)^{\cos(x)} = 1$. These are also the only solutions for which $\cos(x) = 0$. Thus there are $\boxed{2}$ solutions.

16. Compute

$$\sum_{n=1}^{2020} \cos\left(\frac{n\pi}{2020}\right)$$

16. -1

Solution: Since $\cos(x) = -\cos(\pi - x)$, we can pair these solutions to cancel:

$$\cos\left(\frac{n\pi}{2020}\right) + \cos\left(\frac{(2020-n)\pi}{2020}\right) = 0.$$

The only indices which don't have their opposites are $n = 1010$, for which $\cos\left(\frac{1010\pi}{2020}\right) = \cos\left(\frac{\pi}{2}\right) = 0$; and $n = 2020$ (because there is no $n = 0$) for which $\cos\left(\frac{2020\pi}{2020}\right) = \cos(\pi) = -1$. Thus the sum is equal to $\boxed{-1}$.

17. Nikhil likes to mine SchaeferCoin, his favorite cryptocurrency. In order to receive a SchaeferCoin, Nikhil must find a number n such that $n!$ ends in 300 zeroes. Determine the smallest possible value of n.

17. 1210

Solution: The number of zeroes in $n!$ is determined by the number of factors of 5, because for $n \geq 5$ there are always more 2s than 5s. The number of 5s can be found by adding the exponents of 5 in all multiples of 5 less than or equal to n. Guessing and checking, we see that $n = \boxed{1210}$ works because

$$\lfloor 1210/5 \rfloor + \lfloor \lfloor 1210/5 \rfloor /5 \rfloor + \lfloor \lfloor \lfloor 1210/5 \rfloor /5 \rfloor /5 \rfloor + \lfloor \lfloor \lfloor \lfloor 1210/5 \rfloor /5 \rfloor /5 \rfloor /5 \rfloor = 242 + 48 + 9 + 1 = 300,$$

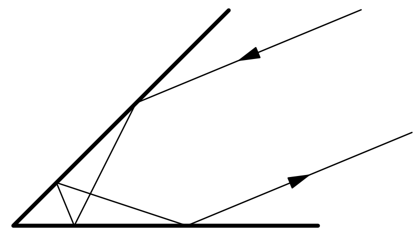
but for any $n < 1210$, there is at least one fewer multiple of 5 less than or equal to n.

18. Stephen has 7 distinct Michigan Math T-Shirts. Every week, Stephen randomly chooses a shirt to wear on Monday, then randomly chooses a different shirt on Tuesday, etc. At the end of the week, Stephen washes all of his shirts and begins again. If there are 4 weeks in a month, determine the probability that Stephen never wears the same shirt two days in a row during the entire month.

18. $\frac{216}{343}$

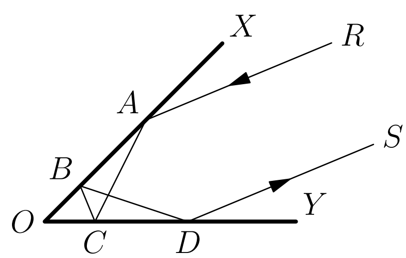
Solution: The only way Stephen could wear the same shirt two days in a row is between a Sunday and a Monday. There are 3 Mondays which come after a Sunday, and for each of them the probability that Stephen picks one of the 6 clean shirts he didn't wear the day before is $\frac{6}{7}$. Since the weeks are independent, the probability Stephen never wears the same shirt two days in a row is $\left(\frac{6}{7}\right)^3 = \boxed{\frac{216}{343}}$.

19. A projectile travels in a straight line towards a wedge-shaped corner between two intersecting walls. Assume that the object is a point, and at a collision, the angle between the trajectory and the wall is the same before and after the collision. If the angle between the first wall and the incoming trajectory is equal to the angle between the second wall and the outgoing trajectory, and the incoming and outgoing trajectories are parallel, then what is the angle between the two walls?



19. $\frac{\pi}{4}$

Solution: By a standard angle chase, we see that the angle between the two walls must be $\pi/4$.



Let $\alpha = \angle BOC$. By the symmetry of the problem, we know that $\angle OBC = \angle OCB$. It follows that both have measure $\frac{\pi-\alpha}{2}$. Thus, $\angle ABD$ and $\angle DCA$ are both of measure $\frac{\pi-\alpha}{2}$. Thus $\angle ABC$ is $\frac{\pi+\alpha}{2}$, and $\angle BCA$ is α. Thus, $\angle BAC$ has measure $\frac{\pi-3\alpha}{2}$. Similarly, $\angle BDC$ has measure $\frac{\pi-3\alpha}{2}$. Finally, we have that $\angle RAX$ and $\angle SDY$ have measure $\frac{\pi-3\alpha}{2}$.

Now, if we draw an imaginary line between A and D, and if we use the fact that $\overline{AR}$ and $\overline{DS}$ are parallel, we see that α is the sum of the measures of $\angle RAX$ and $\angle SDY$, which is $\pi - 3\alpha$. It follows that $\alpha = \boxed{\frac{\pi}{4}}$.

20. Let p_n denote the nth prime number (i.e., $p_1 = 2$). Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that

$$f(p_n x) = p_{n+1} f(x),$$
$$f(1) = 1.$$

Determine $f\left(\frac{2019}{2020}\right)$.

20. $\frac{3385}{6489}$

Solution: Note that the prime factorizations of 2019 and 2020 are $2019 = 3 \cdot 673$ and $2020 = 2^2 \cdot 5 \cdot 101$. Thus

$$f\left(\frac{2019}{2020}\right) = f\left(3 \cdot 673 \cdot \frac{1}{2020}\right) = 5 \cdot 677 f\left(\frac{1}{2020}\right).$$

Also note that

$$\begin{aligned} f\left(\frac{1}{2020}\right) &= \frac{3^2\cdot 7\cdot 103}{3^2\cdot 7\cdot 103} f\left(\frac{1}{2020}\right) \\ &= \frac{1}{3^2\cdot 7\cdot 103} f\left(\frac{2^2\cdot 5\cdot 101}{2020}\right) \\ &= \frac{1}{3^2\cdot 7\cdot 103} f(1) \\ &= \frac{1}{3^2\cdot 7\cdot 103}. \end{aligned}$$

Thus

$$f\left(\frac{2019}{2020}\right) = \frac{5\cdot 677}{3^2\cdot 7\cdot 103} = \boxed{\frac{3385}{6489}}.$$

21. Michigan's football team has the ball in the final minutes of the game! Every play is either first, second, third, or fourth down. On each play, we know that

- If it is first down, there is a $\frac{2}{5}$ chance that the next play will be first down and a $\frac{3}{5}$ chance that the next play will be second down.
- If it is second down, there is a $\frac{1}{2}$ chance that the next play will be first down and a $\frac{1}{2}$ chance that the next play will be third down.
- If it is third down, there is a $\frac{3}{5}$ chance that the next play will be first down and a $\frac{2}{5}$ chance that the next play will be fourth down.
- If it is fourth down, there is a $\frac{1}{10}$ chance the next play will be first down and a $\frac{9}{10}$ chance the game will be over!

Michigan begins on first down. Determine the probability that after 4 plays, Michigan is on first down.

21. $\frac{1039}{2500}$

Solution: The only possible sequences of $4+1=5$ downs resulting in a first down are:

- $1, 2, 3, 4, 1$: Probability $\frac{6}{500}$;
- $1, 2, 3, 1, 1$: Probability $\frac{18}{250}$;
- $1, 2, 1, 2, 1$: Probability $\frac{9}{100}$;
- $1, 2, 1, 1, 1$: Probability $\frac{12}{250}$;
- $1, 1, 2, 3, 1$: Probability $\frac{18}{250}$;
- $1, 1, 2, 1, 1$: Probability $\frac{12}{250}$;
- $1, 1, 1, 2, 1$: Probability $\frac{12}{250}$;
- $1, 1, 1, 1, 1$: Probability $\frac{16}{625}$.

Summing, we have $\boxed{\dfrac{1039}{2500}}$.

Alternate Solution: We write the transition matrix below.

$$M = \begin{pmatrix} \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{1}{10} & 0 \\ \frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{10} & 1 \end{pmatrix}$$

We compute M^5 to be

$$\begin{pmatrix} \frac{1039}{2500} & \frac{97}{250} & \frac{193}{625} & \frac{121}{2500} & 0 \\ \frac{363}{1250} & \frac{111}{500} & \frac{219}{1250} & \frac{69}{2500} & 0 \\ \frac{69}{500} & \frac{3}{20} & \frac{21}{250} & \frac{3}{250} & 0 \\ \frac{6}{125} & \frac{3}{50} & \frac{9}{125} & \frac{3}{250} & 0 \\ \frac{27}{250} & \frac{9}{50} & \frac{9}{25} & \frac{9}{10} & 1 \end{pmatrix}$$

and thus the solution is $\boxed{\dfrac{1039}{2500}}$.

22. Annie and Noah are playing a game with a fair coin. They flip the coin indefinitely. Annie wins if at any point, three consecutive flips are Heads, Heads, Tails. Noah wins if at any point, three consecutive flips are Tails, Heads, Tails. What is the probability Annie wins?

22. $\frac{5}{8}$

Solution: Note that the probability that Annie wins at any given step only depends on the values of the last two flips, because if no one has won yet then the earlier flips will not matter.

Consider the four possibilities for the last two flips $\{TT, TH, HT, HH\}$ We compute the probability that Annie wins for each of these four events. Let

- a denote the probability that Annie wins given TT,
- b denote the probability that Annie wins given TH,
- c denote the probability that Annie wins given HT,
- d denote the probability that Annie wins given HH.

First, we see $a = a/2 + b/2$, because there is no way either player can win after the next flip, and so if Tails is flipped we are back at TT, while if Heads is flipped we are in the case TH.

Second, we see $b = d/2$, because if Tails is flipped Noah wins immediately, while if heads is flipped we are in the situation d.

Third, we see $c = a/2 + b/2$, because there is no way either player can win after the next flip, and so if Tails is flipped we are in the case TT, while if Heads is flipped we are in the case TH.

Fourth, we see $d = 1/2 + d/2$, because if Tails is flipped Annie wins immediately, while if Heads is flipped we are back at HH.

Now we have the system of equations.

$$\begin{aligned} a &= a/2 + b/2 \\ b &= d/2 \\ c &= a/2 + b/2 \\ d &= 1/2 + d/2. \end{aligned}$$

Solving, we find

$$\begin{aligned} a &= 1/2 \\ b &= 1/2 \\ c &= 1/2 \\ d &= 1. \end{aligned}$$

Since the four events $\{TT, TH, HT, HH\}$ each have probability $1/4$, the overall probability that Annie wins is

$$\frac{1/2}{4} + \frac{1/2}{4} + \frac{1/2}{4} + \frac{1}{4} = \boxed{\frac{5}{8}}.$$

23. Matthew wants to paint each of the 6 faces of a wooden die one of red, green, or blue. How many ways are there for him to do this, where two configurations are the same if you can rotate one configuration to the other?

23. 57

Solution: This problem can be solved by applying Burnside's lemma. There are 3^6 total colorings of the faces. Of these,

- 3^6 of them are fixed when doing nothing to the cube;
- 3^3 of them are fixed when doing one of the 6 rotations by $\frac{\pi}{2}$ whose axis is orthogonal to a face;
- 3^4 of them are fixed when doing one of the 3 rotations by π whose axis is orthogonal to a face;
- 3^2 of them are fixed when doing one of the 8 rotations that are along a main diagonal's axis;
- 3^3 of them are fixed when doing one of the 6 rotations whose axis is orthogonal to two opposing edges.

By Burnside's lemma, there are thus

$$\frac{1}{1+6+3+8+6}\left(1 \times 3^6 + 6 \times 3^3 + 3 \times 3^4 + 8 \times 3^2 + 6 \times 3^3\right) = \boxed{57}$$

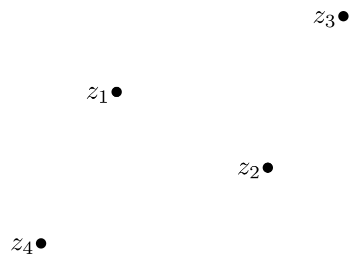
rotationally distinct colorings.

24. There exists a degree 3 polynomial f in four complex variables such that the four complex numbers z_1, z_2, z_3, z_4 form a parallelogram (when thought of as points in $\mathbb{R}^2$) if and only if $f(z_1, z_2, z_3, z_4) = 0$. Find

$f(1,6,1,8)/f(0,3,3,9)$.

24. $\frac{16}{81}$

Solution: We consider four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Without loss of generality, we have the diagram below.

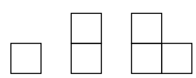


Note that in order for this to form a parallelogram, we must have that $z_1 - z_4 = z_3 - z_2$. Thus, a candidate polynomial would be the polynomial $z_1 + z_2 - z_3 - z_4$, which is 0 when a parallelogram is formed with the points in this configuration. However, we require that this is still satisfied when the points are in a different configuration, so we choose all possible choices of two positive terms and two negative terms in the polynomial. There are $\binom{4}{2} = 6$ of these, but for any polynomial p of this form, $-p$ is also, and so we can discard three of these, and take z_1 to have positive coefficient. Thus, we conclude that the desired polynomial, up to scaling, is given by

$$f(z_1, z_2, z_3, z_4) = (z_1 + z_2 - z_3 - z_4)(z_1 - z_2 + z_3 - z_4)(z_1 - z_2 - z_3 + z_4)$$

We then compute $\frac{(1+6-1-8)(1-6+1-8)(1-6-1+8)}{(0+3-3-9)(0-3+3-9)(0-3-3+9)} = \boxed{\frac{16}{81}}$

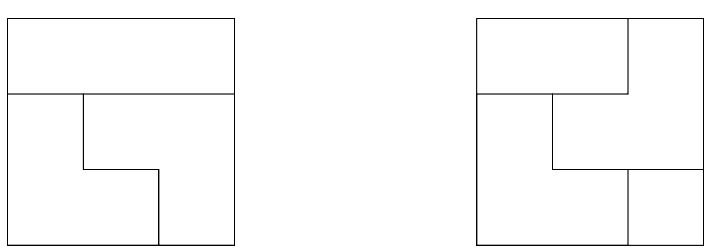
25. Annie's mom bought her a new jigsaw puzzle. The set contains a positioned 3×3 board (in other words, the board cannot be rotated) and an ample supply of tiles of shape:



In how many different ways can she fully fill the board with the tiles if each type of tile must be used at least once?

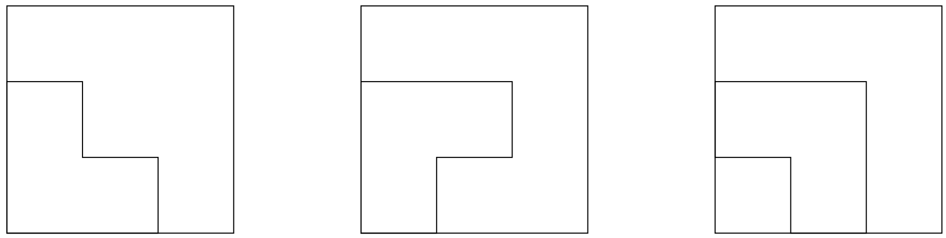
25. 192

Solution: We begin by counting how many tilings use 2 of the 3-square tiles. There are only 2 essentially unique placements (i.e. up to rotation an reflection) that yield possible solutions:



The first configuration can have the bar at the top broken in 2 ways. Since there are 8 rotations an reflections, this yields 16 tilings. The second configuration only yields 8 tilings. Thus, if we use 2 of the 3-square tiles, then there are 24 possible tilings.

Now, suppose we use only 1 of the 3-square tiles. In this case, we consider the following configurations:



The first configuration gives rise to 13 tilings for each of the 4 rotations (to calculate this, condition on how many of the 2-square tiles are used). The second cofiguration gives rise to 11 tilings for each of the 8 rotations and reflections. The third configuration gives rise to 7 tilings for each of the 4 rotations. So, if exactly 1 of the 3-square tiles is used, then there are $13 \times 4 + 11 \times 8 + 7 \times 4 = 168$ tilings.

So, there are $24 + 168 = \boxed{192}$ total tilings.

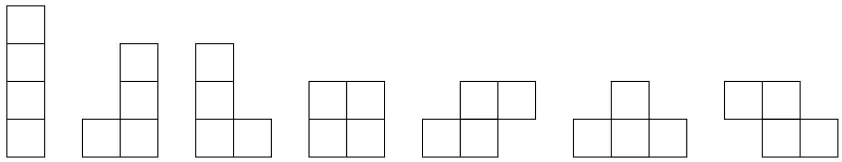
26. Define a recursive sequence by $a_0 = 1$, $a_1 = 1$, and $a_n = a_{n-1}/3 + a_{n-2}/4$ for $n \geq 2$. Find $\sum_{n=0}^{\infty} a_n$.

26. 4

Solution: We find

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= 1 + 1 + \sum_{n=2}^{\infty} a_n \\
&= 1 + 1 + \sum_{n=2}^{\infty} \left(\frac{1}{3} a_{n-1} + \frac{1}{4} a_{n-2} \right) \\
&= 1 + 1 + \frac{1}{3} \sum_{n=2}^{\infty} a_{n-1} + \frac{1}{4} \sum_{n=2}^{\infty} a_{n-2} \\
&= 1 + 1 + \frac{1}{3} \sum_{n=1}^{\infty} a_n + \frac{1}{4} \sum_{n=0}^{\infty} a_n \\
&= 1 + 1 - \frac{1}{3} + \frac{1}{3} \sum_{n=0}^{\infty} a_n + \frac{1}{4} \sum_{n=0}^{\infty} a_n \\
&= \frac{5}{3} + \frac{7}{12} \sum_{n=0}^{\infty} a_n \\
\frac{5}{12} \sum_{n=0}^{\infty} a_n &= \frac{5}{3} \\
\sum_{n=0}^{\infty} a_n &= \boxed{4}.
\end{aligned}$$

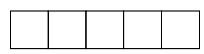
27. An n-polyomino is a 2-dimensional polygon made up of n 1×1 squares, such that every square touches any other square at only edges and corners. For example, all 4-polyominos are shown below.



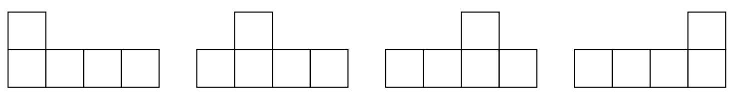
How many possible 5-polyominoes exist, if two polyominoes are the same if one is a rotation (but not reflection) of another?

27. 18

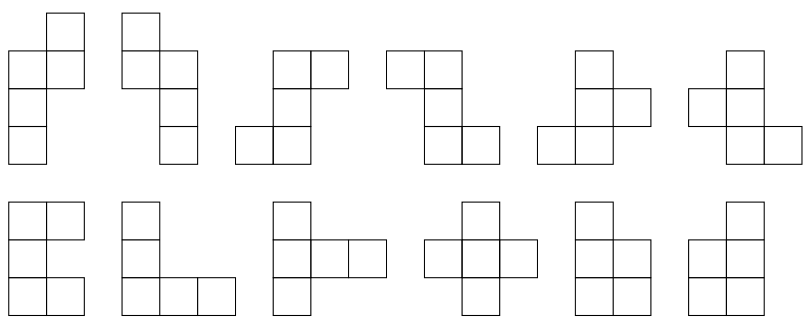
Solution: We consider casewise by the longest straight segment of blocks. Clearly, there is one piece with 5 blocks in a row.



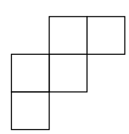
With four pieces in a row, we can vary where the fifth block is placed, which results in the following 4 configurations.



With three pieces in a row, we have 12 pieces, obtained by varying where the remaining two pieces go.



Finally, for two pieces in a row, there is exactly one piece.



We conclude that there are $1 + 4 + 12 + 1 = \boxed{18}$ total pieces.

28. Noah rolls three six-sided dice, numbered from 1 to 6. What is the probability that the numbers which come up can be the side lengths of a triangle?

28. $\frac{37}{72}$

Solution: If one of the dice shows a 1, then the other two dice must have the same value. There is a probability of $\frac{16}{216}$ of this happening.

Now suppose no die shows a 1. If one of the dice shows a 2, then the remaining two dice must differ by at most 1. There is a probability of $\frac{34}{216}$ of this happening.

Now suppose no die shows a 1 or a 2. Then the only way that the dice rolled cannot form a triangle is if we roll two 3s and a 6. So the probability that we can form a triangle in this case is $\frac{61}{216}$.

So, the probability that we can form a triangle is $\frac{16}{216} + \frac{34}{216} + \frac{61}{216} = \boxed{\dfrac{37}{72}}$.

Alternate Solution: Let the first die's number be denoted by a, the second die's number by b, and the third die's number by c. Then the set of possible outcomes (a, b, c) for the three dice can be associated with lattice points in a cube.

If (a, b, c) are the sides of a nondegenerate triangle, then the triangle inequality gives us three inequalities:

$$\begin{aligned} a &< b + c \\ b &< a + c \\ c &< a + b. \end{aligned}$$

Each of these removes a right tetrahedron from corner of the cube with 5 points along each sides. These tetrahedra are disjoint, and each contains

$$15 + 10 + 6 + 3 + 1 = 35$$

lattice points. Therefore the number of outcomes for which (a, b, c) are the sides of a triangle is

$$6^3 - 3 \cdot 35 = 216 - 105 = 111$$

So the desired probability is $\frac{111}{216} = \boxed{\dfrac{37}{72}}$.

29. Let x, y, z be non-negative real numbers such that $x + y + z = 1$.

$$(x + 5y + 7z)\left(x + \frac{y}{5} + \frac{z}{7}\right)$$

Let M denote the maximum of the expression above, and let N denote the minimum. Determine $M + N$.

29. $\frac{23}{7}$

Solution: First we note that, by the Cauchy-Schwarz inequality,

$$\begin{aligned} 1 &= (x + y + z)^2 \\ &= \left(\sqrt{x} \cdot \sqrt{x} + \sqrt{5y} \cdot \sqrt{\frac{y}{5}} + \sqrt{7z} \cdot \sqrt{\frac{z}{7}}\right)^2 \\ &\le (x + 5y + 7z)\left(x + \frac{y}{5} + \frac{z}{7}\right) \end{aligned}$$

Furthermore, this lower bound is achieved when $x = 1$, $y = 0$, and $z = 0$. So $N = 1$.

For the upper bound, we use the AM-GM inequality,

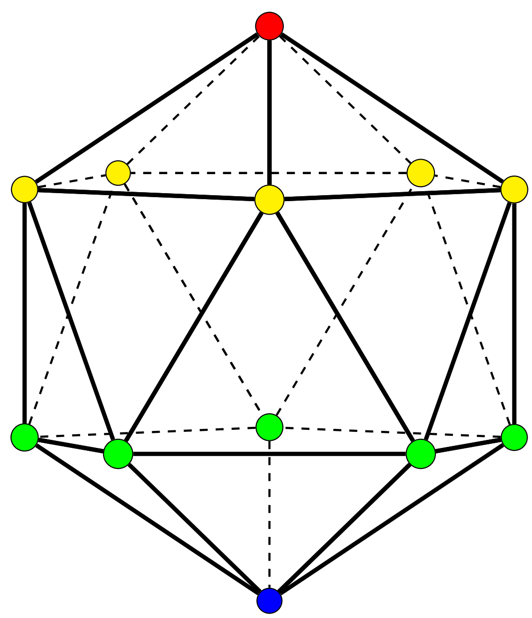
$$\begin{aligned}(x+5y+7z)\left(x+\frac{y}{5}+\frac{z}{7}\right) &= \frac{1}{7}(x+5y+7z)\left(7x+\frac{7}{5}y+z\right)\\ &\le \frac{1}{7}\cdot\frac{1}{4}\left((x+5y+7z)+\left(7x+\frac{7}{5}y+z\right)\right)^2\\ &= \frac{1}{28}\left(8x+\frac{32}{5}y+8z\right)^2\\ &\le \frac{1}{28}\left(8x+8y+8z\right)^2\\ &= \frac{16}{7}\end{aligned}$$

The upper bound is achieved when $x=\frac{1}{2}, y=0$, and $z=\frac{1}{2}$, so $M=\frac{16}{7}$, which gives $M+N=\frac{23}{7}$.

30. Ani the Ant is on a vertex of a regular icosahedron with side length 2. Every second, Ani moves to one of the adjacent vertices with equal probability. After 3 seconds, what is the expected value of Ani's straight-line distance from his original location?

30. $\frac{34}{25}+\frac{8}{25}\sqrt{5}+\frac{2}{25}\sqrt{10+2\sqrt{5}}$

Solution: Consider the rendering below. We can choose coordinates for the icosahedron to be $(0,\pm1,\pm\varphi)$, $(\pm1,\pm\varphi,0)$, and $(\pm\varphi,0,\pm1)$.



Fix a starting vertex v, and without loss of generality, let v be the topmost vertex. We first compute the expected number of edges away from v. We classify points on the icosahedron as follows.

1. The vertex v, colored red.

2. The 5 vertices exactly one edge away from v, colored yellow.

3. The 5 vertices exactly two edges away from v, colored green.

4. The vertex antipodal to v, colored *blue*.

Note that at any vertex, we have 5 possible moves, so there are 125 total possible paths. We enumerate the number of ways in which we arrive at each color in three steps.

1. To reach the red vertex, we must go to a yellow vertex, move to an adjacent yellow vertex, and return to the red vertex. There are 5 ways to choose the initial yellow vertex, then 2 ways to choose the adjacent yellow vertex, for a total of 10 ways.

2. To reach a yellow vertex, we can do one of three things.

   (a) We can start at the red vertex, move to a yellow vertex, back to the red vertex, and back to a yellow vertex. There are 5 choices of initial red vertex, and 5 choices of the second red vertex, for 25 total paths.

   (b) We can start at the red vertex, move to a yellow vertex, down to a green vertex, and back up to a yellow vertex. There are 5 ways to choose the initial yellow vertex, 2 ways to choose the next green vertex, and 2 ways to come back up to a yellow vertex, for a total of 20 total paths.

   (c) We can start at the red vertex, move to a yellow vertex, and move to an adjacent yellow vertex twice. There are 5 ways to choose an initial yellow vertex, and then 2 choices for the second yellow vertex, and 2 choices for the third yellow vertex, for a total of 20 total paths.

   Thus, we have 65 total paths to reach a yellow vertex.

3. To reach a green vertex, we can do one of two things.

   (a) We can start at the red vertex, move to a yellow vertex, move to another yellow vertex, and then move to a green vertex. We have 5 choices for the initial yellow vertex, 2 choices for the second yellow vertex, and 2 choices for the green vertex, for 20 total paths.

   (b) We can start at the red vertex, move to a yellow vertex, move to a green vertex, and move to a green vertex. There are 5 ways to choose a yellow vertex, 2 ways to choose a green vertex, and 2 ways to choose a green vertex, for 20 total paths.

   Thus, we have 40 total paths.

4. To reach the blue vertex, we must reach one vertex of every color. There are 5 ways to choose a yellow vertex, 2 ways to choose a green vertex, and 1 way to choose the blue vertex, for a total of 10 paths.

We verify that $10 + 65 + 40 + 10 = 125$, and thus these are all possible paths.

We then compute distances. We can compute distances by, without loss of generality, comparing

$(1, 0, \varphi)$ to every choice of $\pm$ in $(\pm 1, 0, \pm\varphi)$. We compute

$$\begin{aligned}
(1, 0, \varphi) \to (1, 0, \varphi) &= \sqrt{0 + 0 + 0} = 0 \\
(1, 0, \varphi) \to (-1, 0, \varphi) &= \sqrt{4 + 0 + 0} = 2 \\
(1, 0, \varphi) \to (1, 0, -\varphi) &= \sqrt{(2\varphi)^2} = 2\varphi = 1 + \sqrt{5} \\
(1, 0, \varphi) \to (-1, 0, -\varphi) &= \sqrt{4 + 4\varphi^2} \\
&= \sqrt{4 + (1 + \sqrt{5})^2} \\
&= \sqrt{10 + 2\sqrt{5}}
\end{aligned}$$

Clearly, these must be ordered in increasing order, and so we assign distances as follows.

1. The distance from the red vertex to a red vertex is 0.
2. The distance from a red vertex to a yellow vertex is 2.
3. The distance from a red vertex to a green vertex is $1 + \sqrt{5}$.
4. The distance from a red vertex to a blue vertex is $\sqrt{10 + 2\sqrt{5}}$.

We then compute expected value. We have that the expected value is given by

$$\begin{aligned}
E &= \frac{10}{125}(0) + \frac{65}{125}(2) + \frac{40}{125}\left(1 + \sqrt{5}\right) + \frac{10}{125}\left(\sqrt{10 + 2\sqrt{5}}\right) \\
&= \frac{130}{125} + \frac{40}{125} + \frac{40}{125}\sqrt{5} + \frac{10}{125}\left(\sqrt{10 + 2\sqrt{5}}\right) \\
&= \frac{170}{125} + \frac{40}{125}\sqrt{5} + \frac{10}{125}\sqrt{10 + 2\sqrt{5}} \\
&= \boxed{\frac{34}{25} + \frac{8}{25}\sqrt{5} + \frac{2}{25}\sqrt{10 + 2\sqrt{5}}}
\end{aligned}$$

31. **(TIEBREAKER)** How many total lines of LaTeX were used in typing the Wolverine Round? Ties will be broken by the closest answer without going over.

31. **871**

Solution: Thank you for attending Lemma 2020!