Econ 609/610

1.9 Book: Christian Gollier, Economics of Time and Risk mkimball@umich.edu

Read Chapters 1-6 by 1 week from Wednesday (1.18)

I. One and two-period models
   A. Diffidence Theorem
   B. Method of Polars
II. Multi-period models (Basic)
   A. Backwards Recursion (Programming) (Induction)

   a lot of work characterizing the value function
   $V(K_t)$ = function giving lifetime utility as a function of the current vector of state variables (anything that can only change gradually over time)

   B. Symmetry Theorem
III. Multi-period models (Advanced)
   A. Horizontal and Vertical Aggregation (Conjugate functions)
   B. Preser-Max Theorem
IV. Computation

When people do wiered things

Paradox of Choice

(1) Look for deep wisdom

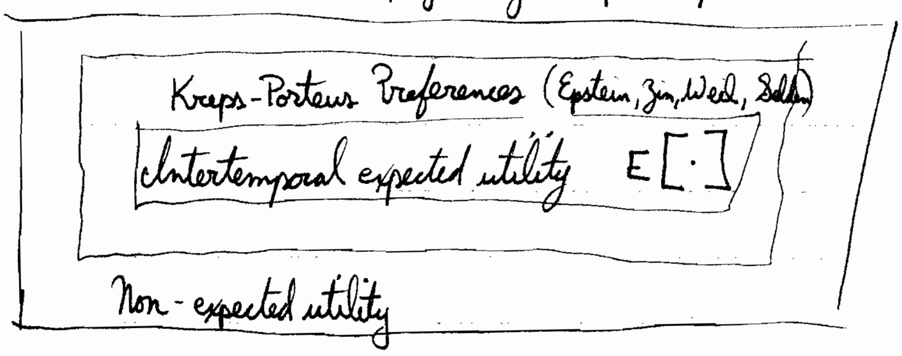
(2) Look at bounded cognition (people aren't infinitely smart)

(3) ↳ (1) Think about how hard to think about something (infinite regress prob.)

(2) Approximate $IQ < \infty$ by information transmission limitations

(3) Agent-based modelling → Scott Page

(3) Look at other psychological principles such as evolved emotions



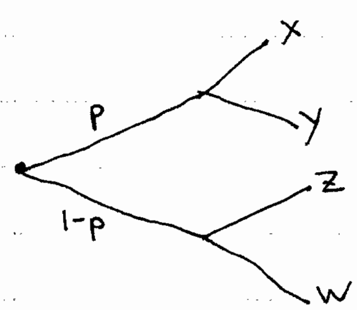
Key Principles to get expected utility (these are normative; people may violate them)

(1) Fungibility (consequentialism) (labelling doesn't matter)

(2) Transitivity

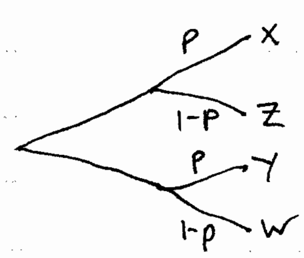
(3) Sunk cost principle

Independence Axiom



$X \succsim Y \quad X = (P_1, P_2, 1-P_1-P_2)$

$Z \succsim W$



$\Rightarrow pX + (1-p)Z \succsim pY + (1-p)W$

1.11 Proof of expected utility theorem

W = worst

B = best

$U(W) = 0$

$U(B) = 1$

By continuity, $\exists\, q_X, q_Y$, $X \sim q_X B + (1-q_X) W$

$Y \sim q_Y B + (1-q_Y) W$

Claim: We can take $U(X) = q_X$ and $U(Y) = q_Y$

(Utility is a probability metric here)

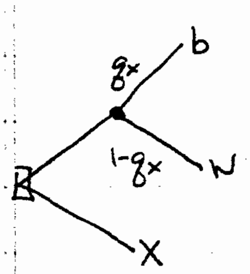
To show:

$pX + (1-p)Y \sim p q_X + (1-p) q_Y = p\,U(X) + (1-p)\,U(Y)$

Step 1: $pX + (1-p)Y \sim p q_X b + p(1-q_X) W + (1-p)Y$

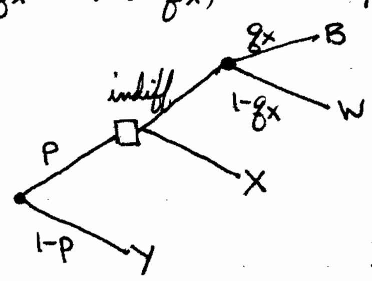
fungibility

indiff.



Version of sunk cost principle $\Longleftrightarrow$

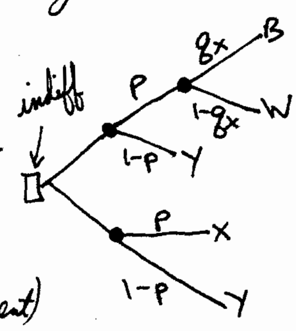
indiff



Equivalence of dif. representations $\Longleftrightarrow$

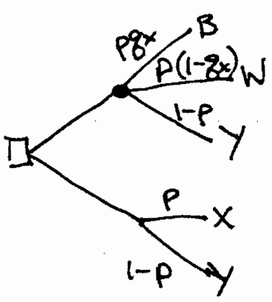
(same choice if given as written instructions to agent)

indiff

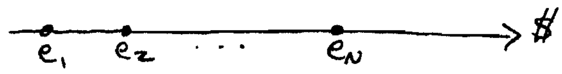


Compound Probability equivalence $\Longleftrightarrow$

(Consequentialism)



$(pq_X + (1-p)q_Y)B + (p(1-q_X) + (1-p)(1-q_Y))W$

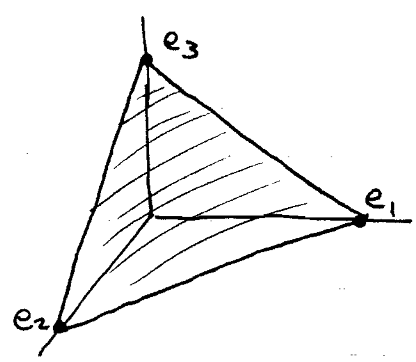


$E = (e_1, e_2, \cdots, e_N)'$

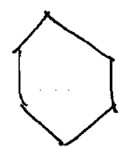
Y is a random variable with prob. $\underline{P_Y} = (P_{Y,1}; P_{Y,2}; \cdots; P_{Y,N})'$
Z $\quad \underline{P_Z} = (P_{Z,1}; P_{Z,2}; \ldots; P_{Z,N})'$

$\underline{Q} = \underline{P_Y} - \underline{P_Z} = (P_{Y,1} - P_{Z,1}; P_{Y,2} - P_{Z,2}; \cdots; P_{Y,N} - P_{Z,N})'$
$\underline{1} = (1, 1, \ldots, 1)'$

$\underline{1} \cdot \underline{P_Y} = 1 \Leftrightarrow \sum_{i=1}^{N} P_{Y,i} = 1$
$\underline{1} \cdot \underline{P_Z} = 1$
$\underline{1} \cdot Q = 0$



Q lives on a hexagon
$(-1, 1, 0)$ $(1, -1, 0)$ $(1, 0, -1)$ $(-1, 0, 1)$
$(0, 1, -1)$ $(0, -1, 1)$ vertices



$U = (U_1 \; U_2 \cdots U_N)'$ $\quad U_i = \mathcal{U}(e_i)$
$\underline{U} \cdot \underline{Q} = \underline{U} \cdot (\underline{P_Y} - \underline{P_Z}) \geq 0 \Rightarrow$ choose Y $\qquad \underline{U} \cdot \underline{Q} \leq 0 \Rightarrow$ choose Z
$U_1 P_{Y,1} + U_2 P_{Y,2} + \cdots + U_N P_{Y,N} - U_1 P_{Z,1} - U_2 P_{Z,2} - \cdots - U_N P_{Z,N}$

From Convex Analysis,
$\mathcal{Q}$ = set of Q's
$\mathcal{U}$ = set of U's

$\mathcal{U} = \mathcal{Q}^{\circ}$ (Polar of Q, Q polar) $= \{U \mid U \cdot Q \geq 0 \;\; \forall Q \in \mathcal{Q}\}$
set of all utility functions for which you would choose Y over Z when $P_Y - P_Z \in \mathcal{Q}$.

Alternatively, $\mathcal{Q} = \mathcal{U}^\circ = \{ \underline{Q} \mid \underline{U}\cdot\underline{Q} \geq 0 \ \forall \underline{U} \in \mathcal{U} \}$

set of random variable pairs for which Y is chosen over Z for all utilities function is set $\mathcal{U}$.

I. Find $\mathcal{U}^\circ$ if $\mathcal{U}$ = increasing functions

= increasing concave functions

expectation operator

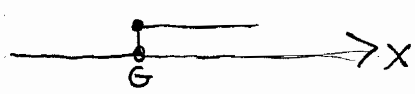
II Find $\mathcal{Q}^\circ$ if $\mathcal{Q} = \{ \underline{P_Y} - \underline{P_Z} \mid P_Y = \delta(\omega), E\cdot(\underline{P_Y} - \underline{P_Z}) = 0$

$$\delta = \left[\begin{array}{c}0\\0\\0\\\vdots\\1\\0\\0\end{array}\right]\begin{array}{l}\\ \\ \\ \\ \leftarrow \omega\\ \\ \\ \end{array}$$

$Z = Y + \tilde{\varepsilon}$, $E[\tilde{\varepsilon}] = 0$

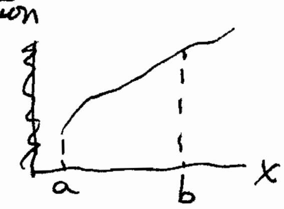
$\mathcal{Q} = \{ \underline{P_Y} - \underline{P_Z} \mid \underline{P_Y} = \delta(\omega), u_1\cdot(\underline{P_Y} - \underline{P_Z}) \geq 0 \}$ set of rv Z worse than constant w.

Representing the set of increasing functions



indicator function

$1(X \geq \theta)$



$$u(x) = u(a) + \int_a^b u'(\theta)\, 1(x \geq \theta)\, d\theta$$
$$= u(a) + \int_a^b 1(x \geq \theta)\, du(\theta)$$

Scaling this, we can get $u(x) = E_\theta[1(x \geq \theta)]$

$\mathcal{U}$ = non-negative linear combinations of the extremal or basis functions

If $\mathcal{U}$ = set of increasing functions, $\mathcal{U}^0 = \{ 1(x \geq \theta) \}$

if $\underline{U} \cdot \underline{Q} \geq 0$ $\forall$ inc. functions

$\Rightarrow \underline{U} \cdot 1(x \geq \theta) \geq 0 \quad \forall \theta$

If all increasing functions prefer Y to Z, step function must prefer Y to Z
⇒ necessary condition for first-order stochastic dominance.
If all step functions $1(x \geq \theta)$ prefer Y to Z, then all increasing functions prefer Y to Z.

$$\cancel{X} = \begin{bmatrix} e_1 = x_1 \\ e_2 = x_2 \\ \vdots \end{bmatrix}$$

$$\left[U(a) + \int_a^b 1(x \geq \theta) \, dU(\theta) \right] \cdot P_Y - P_Z$$

$F(\theta)$

θ

$$E_\theta \left[1(x \geq \theta) \right] = E_U \left[1(x \geq \theta) \right]$$

$$E_{F_Y} \left[E_U \left[1(x \geq \theta) \right] \right] - E_{F_Z} \left[E_U \left[1(x \geq \theta) \right] \right]$$

$$E_U \left[E_{F_Y} \left[1(x \geq \theta) \right] \} - E_{F_Z} \left[1(x \geq \theta) \right] \right] = E_U \left[\left[1 - F_Y(\theta) \right] - \left[1 - F_Z(\theta) \right] \right]$$

$$= E_U \left[\underbrace{F_Z(\theta) - F_Y(\theta)}_{\text{need this to be } \geq 0} \right] \geq 0$$

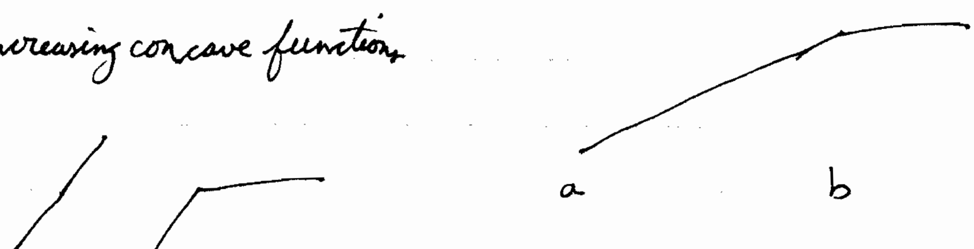
4

b_1 basis function $\quad Q \cdot b_1 \geq 0$

b_2 basis function $\quad Q \cdot b_2 \geq 0$

$\Rightarrow Q(\alpha b_1 + (1-\alpha) b_2) \geq 0$ for all $\alpha \in [0,1]$

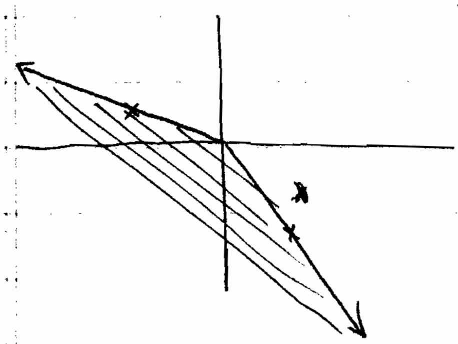
Increasing concave functions



$\min(X - \theta, 0)$

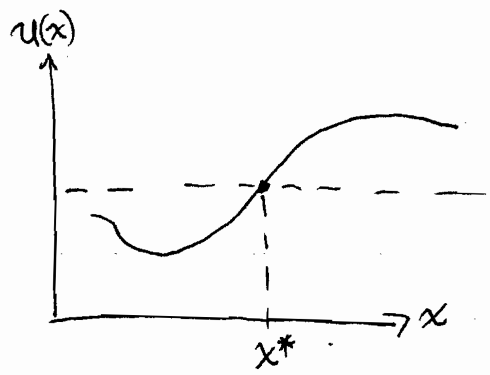
$$u(x) = u(b) - \int_a^b u''(\theta) \min(x-\theta, 0)\, d\theta$$

$(\mathcal{U}^\circ)^\circ = ccc(\mathcal{U})$ closed convex cone of $\mathcal{U}$

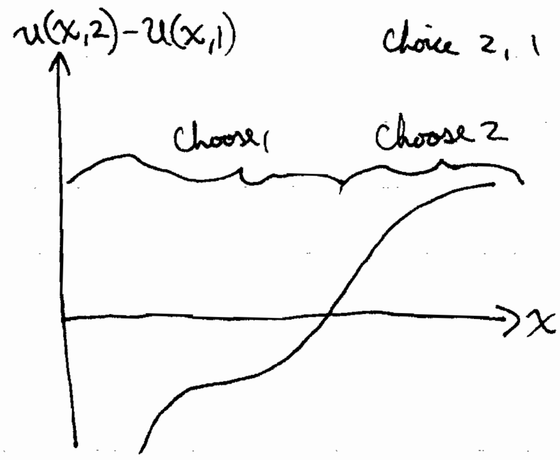


cone → rays from origin

1.18



Choose ~~up~~ go ↗
stay →

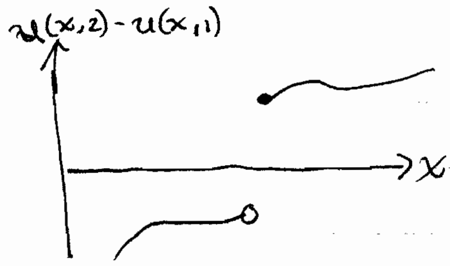


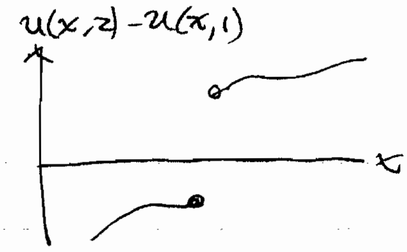
choice 2, 1

single crossing property

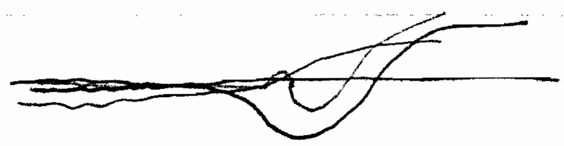
↑ x doesn't reverse your choice after x^*,

SCP necessary to get monotone comparative statics results





Is the set of functions satisfying SCP convex? NO



If, however, we fix x^*, and take functions that have SCP at x^*, that set is convex.

$\mathcal{Q} = \{ Q \mid Q = P - 1(x = x^*),\ \exists_P \in \Delta \}$ changing from being @ x^* to some other dist.

Interested in dist better.

$\mathcal{U} = \{ u \mid u(x) \geq u(x^*),\ \forall x > x^*;\ u(x) \leq u(x^*)\ \forall x < x^* \}$

Basis functions? Constant function

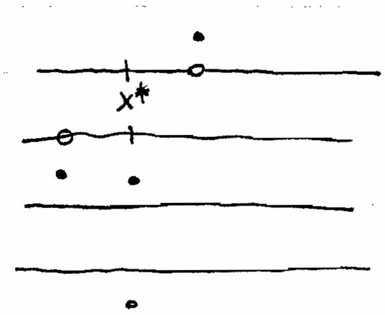
$b_\theta = 1(x=\theta)$, $\theta > x^*$

$b_\theta = -1(x=\theta)$, $\theta < x^*$

$b_{x^*_+} = 1(x=x^*)$

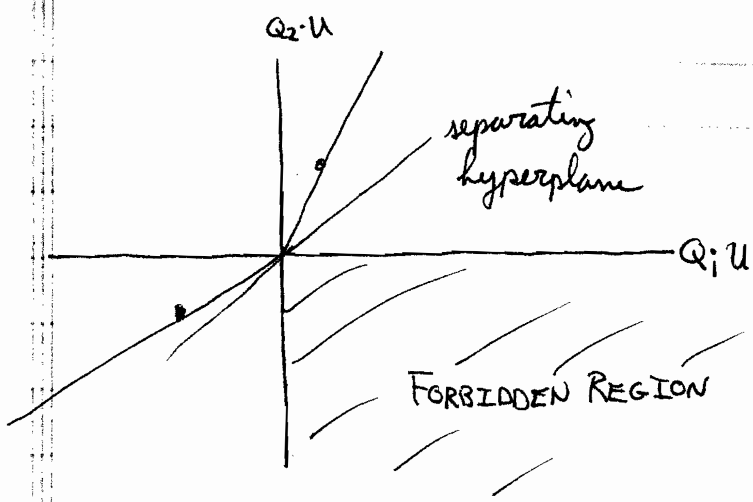
$b_{x^*_-} = -1(x=x^*)$

$b_{cons} = 1$



~~If 0~~ for all $u \in \mathcal{U}$, if $Q_1 \cdot u \geq 0$, then $Q_2 \cdot u \geq 0$.

$P_1 \succsim x^* \Rightarrow P_2 \succsim x^*$



combinations $(Q_1 \cdot u, Q_2 \cdot u)$ is convex set

if interiors of two sets disjoint, $\exists$ a separating hyperplane.

$\exists m_{\geq 0}\ Q_2 \cdot u \geq m Q_1 \cdot u \iff \exists m \geq 0$ s.t. $Q_2 \cdot b_\theta \geq m Q_1 \cdot b_\theta$ for all $b_\theta \in$ whole set of basis funct

$[P_2 - 1(\underset{x=x^*}{x^*})]1(x=\theta) \geq m \quad [P_1 - 1(x^*)]1(x=\theta) \quad \forall \theta > x^*$

$[P_2 - 1(x^*)][-1(x=\theta)] \geq m \quad [P_1 - 1(x=x^*)][-1(x=\theta)] \quad \forall \theta < x^*$

$f_2(\theta) \geq m f_1(\theta) \quad \forall \theta > x^*$

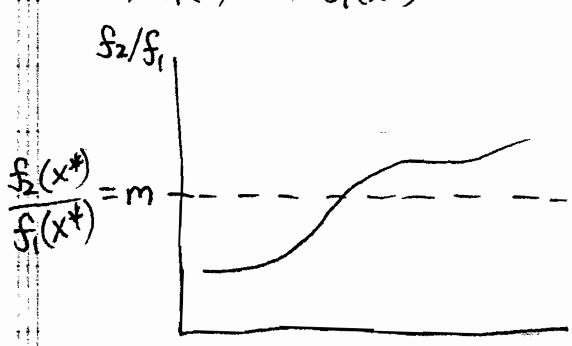
$-f_2(\theta) \geq -m f_1(\theta) \quad \forall \theta < x^* \qquad f_2(\theta) \leq m f_1(\theta) \quad \forall \theta < x^*$

$f_2(x)/f_1(x) \gtrless m$ as $x \gtrless x^*$

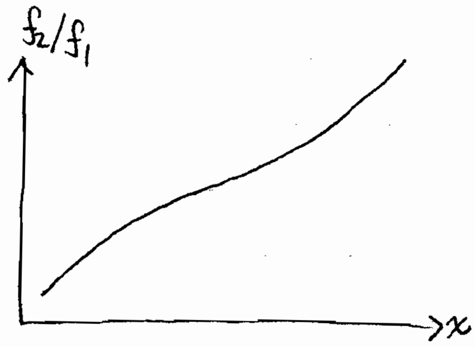
$f_2(x^*) \geq m f_1(x^*)$ and $f_2(x^*) \leq m f_1(x^*) \Rightarrow f_2(x^*) = m f_1(x^*)$

$\Rightarrow m = f_2(x^*)/f_1(x^*)$

$f_2(x)/f_1(x) \gtrless \dfrac{f_2(x^*)}{f_1(x^*)}$ as $x \gtrless x^*$



what conditions will make this hold for any x^*? Monotonicity



We're increasing set of utility functions but decreasing set of pairs of pairs (P_1, P_2).

log-supermodularity

$$\log(f_2(x)/f_1(x)) = \log(f_2(x)) - \log(f_1(x))$$

if $\geq 0 \Rightarrow$ log supermodularity

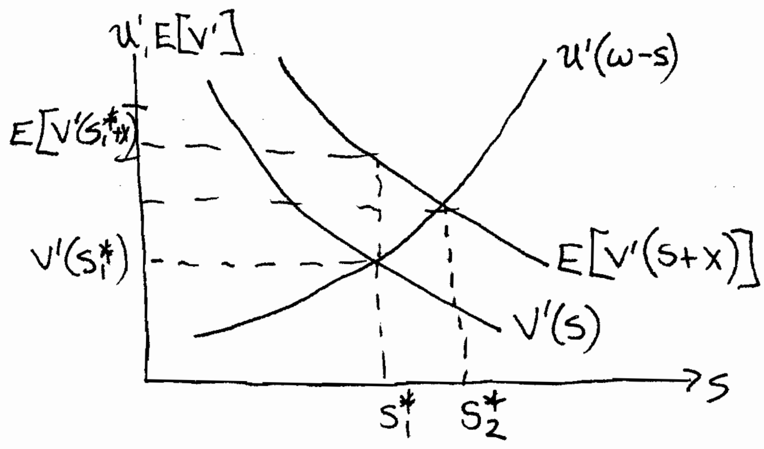
$\cancel{X}$	X	$E[X] = 0$	
$\cancel{u(x)}$	$V(\omega+X)$	$E[V(\omega+\tilde{X})] - V(\omega) \leq 0$	You don't like $\tilde{X}$
$\cancel{u'(x)}$	$V'(\omega+X)$	$E[V'(\omega+\tilde{X})] - V'(\omega) \geq 0$	
	$XV''(\omega+X)$		

② $\max_s \; u(\omega-s) + E[V(s+x)]$

① $\max_s \; u(\omega-s) + v(s)$

$$u'(\omega-s) = E[V'(s+x)]$$

$$u'(\omega-s) = v'(s)$$



what does value function have to look like to get this

$E[\tilde{X}V'(\omega+\tilde{x})] \geq 0$ Portfolio allocation

$$\max_{\alpha} E[u(\omega+\alpha\tilde{x})]$$

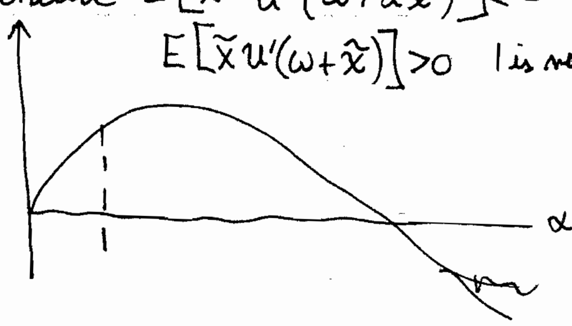
FONC $\quad E\left[\frac{\partial}{\partial\alpha} u(\omega+\alpha\tilde{x})\right] = 0$

$\Leftrightarrow \quad E[\tilde{x}u'(\omega+\alpha^*\tilde{x})] = 0$

~~Unit of $\tilde{x}$ is 2~~ α^* is right amount of $\tilde{x}$.

If $u(\cdot)$ concave $E[\tilde{x}^2 u''(\omega+\alpha\tilde{x})] \leq 0$

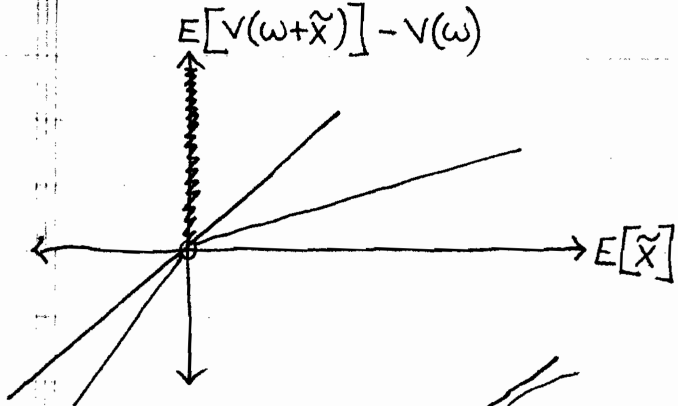
$E[\tilde{x}u'(\omega+\tilde{x})] > 0$ 1 is not enough



	X	$V(\omega+\tilde{x})$	$V'(\omega+\tilde{x})$	$\tilde{x}V'(\omega+\tilde{x})$
X				
$V(\omega+\tilde{x})$				
$V'(\omega+\tilde{x})$				
$XV'(\omega+\tilde{x})$				

if $E[\tilde{x}] = 0$, $E[V(\omega+\tilde{x})] - V(\omega) \leq 0$ disliking mean zero r.v.'s.

$V(\cdot)$ is diffident at ω (dislikes mean zero r.v.'s at particular ω)

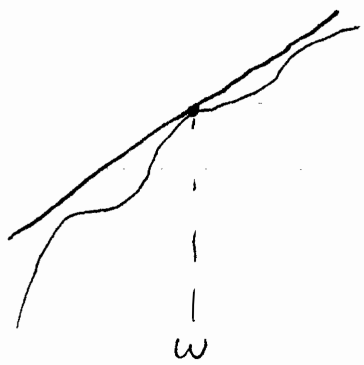


$\exists m$

$E[V(\omega+\tilde{x})] - V(\omega) \lesseqgtr m E[\tilde{x}]$

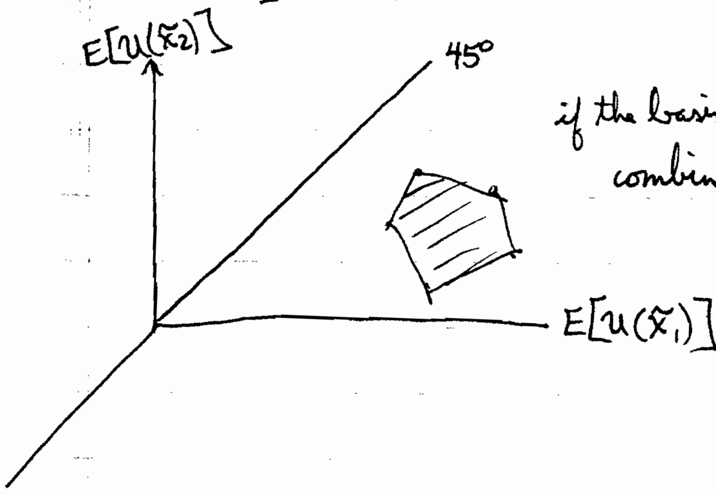
$\Updownarrow$

$\exists m \quad E[V(\omega+x)] - V(\omega) \lesseqgtr m x$



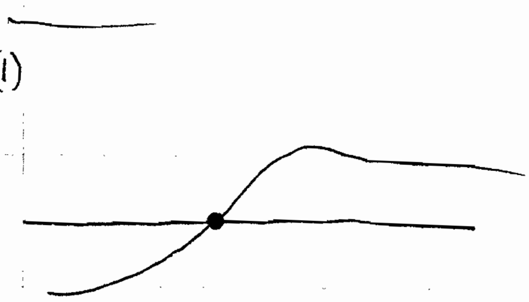
1.23 $E[u(\tilde{x}_2)] \lesseqgtr E[u(\tilde{x}_1)]$

Check 2004 notes



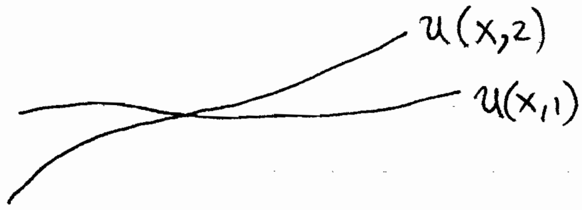
if the basis functions are all below, the convex combination of them is too.

(1)

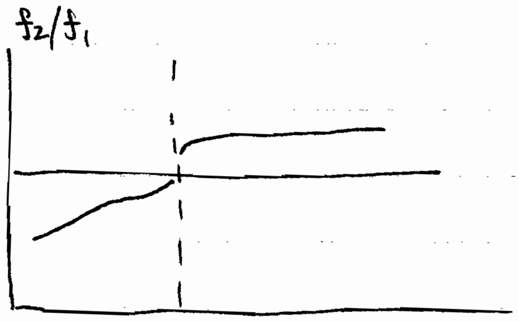


comparing @ crossing w/certainty vs around it with density $f(\cdot)$

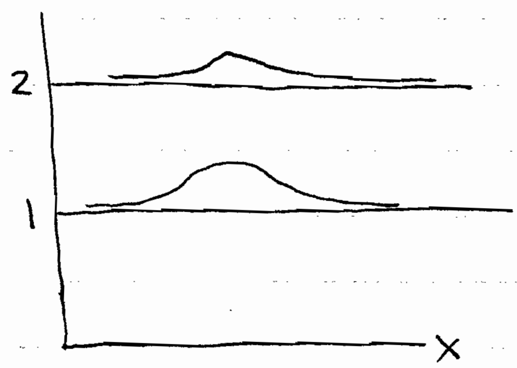
(2)



(1)

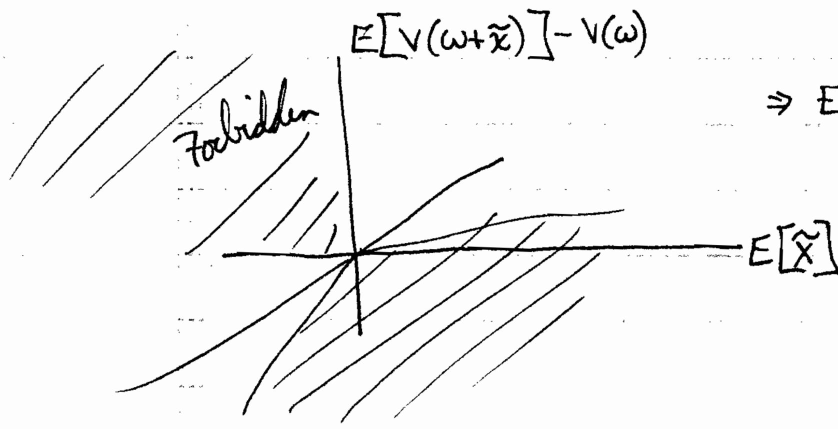


(2)



$\forall \tilde{x}$ if $E[\tilde{x}] \leqslant 0$, then $E[\check{v}(\omega+\tilde{x})] - \check{v}(\omega) \leqslant 0$

For what $\check{v}$'s will this be true



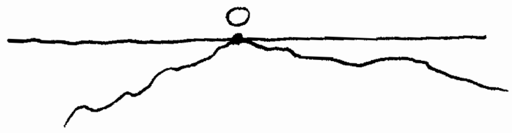
$$\Rightarrow E[v(\omega+\tilde{x})] - v(\omega) \leqslant m E[\tilde{x}]$$

$$\Downarrow$$

$$v(\omega+x) + v(\omega) \leqslant mx$$ by extremal degenerate pdf $= 1$ @ x

$\phi(x)$: $V(\omega+x) - V(\omega) - mx \leq 0$

$\phi(x) \leq 0 \quad , \phi(0) = 0 \qquad \Rightarrow \phi'(0) = 0$ if $\phi(\cdot)$ is differentiable



$\Rightarrow V'(\omega+x) - m \Big|_{x=0} = V'(x) - m = 0$

$\Rightarrow \phi(x) = V(\omega+x) - V(\omega) - V'(\omega) x \leq 0$ necessary & sufficient condition

sufficiency: $V(\omega+x) - V(\omega) \leq V'(\omega)\, x$

$\Rightarrow E\left[V(\omega+\tilde{x}) - V(\omega)\right] \leq E\left[V'(\omega)\tilde{x}\right]$

$= E[\tilde{x}]\,V'(\omega) \leq 0$ for $E[\tilde{x}] \leq 0$

$\forall \omega, \tilde{x}$ if $E[\tilde{x}] \leq 0$, then $E\left[V(\omega+\tilde{x})\right] - V(\omega) \leq 0$

$\forall \omega \Rightarrow \phi(x) = V(\omega+x) - V(\omega) - V'(\omega) x \leq 0$ central condition (nsc) holding globally

$\Longrightarrow \phi''(0) \leq 0$

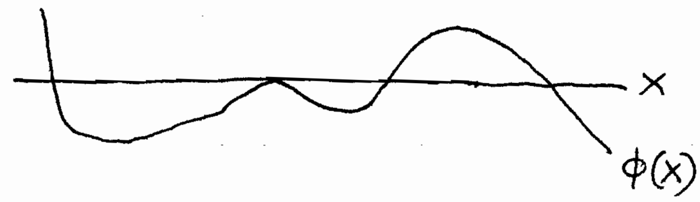
$V''(\omega+x)\Big|_{x=0} = V''(\omega) \leq 0$ this is necessary (local condition) holding globally

$V'(\omega) = m \quad V'(\omega+x)\Big|_{x=0} = m\,\Big|_{x=0}$

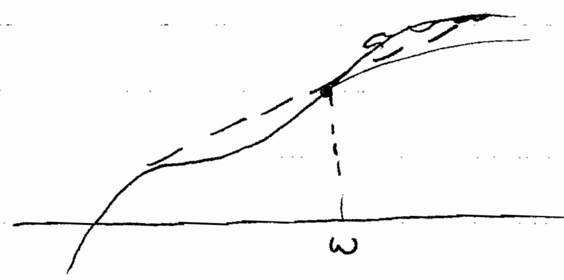
$V''(\omega+x)\Big|_{x=0} \leq 0\,\Big|_{x=0}$

Question: if the local necessary condition holds globally, ~~is it equivalent to~~ does it imply the central necessary & sufficient condition holds globally ($\forall \omega$)? sometimes

We need to rule out



Knowing $\phi(\cdot)$ concave, is there way to rule out $\phi(x) = V(\omega+x) - V(\omega) - V'(\omega)x \leq 0$



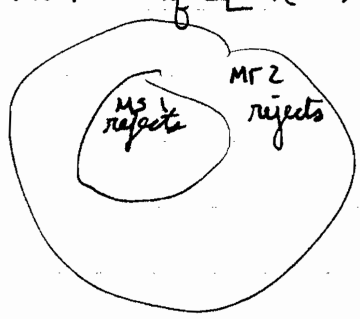
$V(\omega+x) \leq V(\omega) + V'(\omega)x$ ← tangent line

Concavity is enough to show this

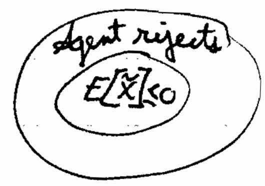
$$V(\omega+x) - V(\omega) - V'(\omega)x = \int_0^x \int_0^{\zeta} V''(\omega+\xi)\,d\xi\,d\zeta$$
$$= \int_0^x \left[V'(\omega+\zeta) - V'(\omega)\right] d\zeta$$

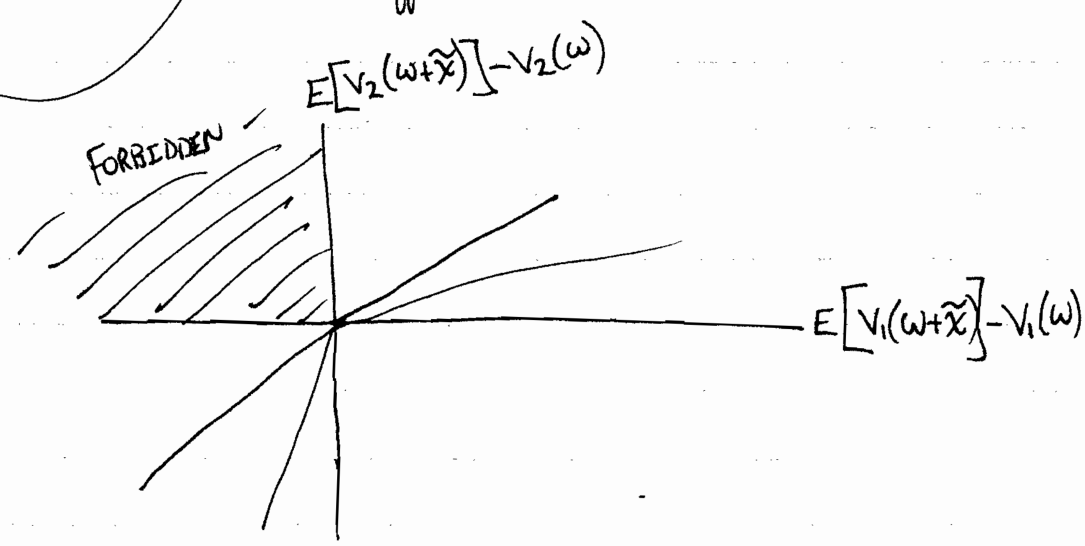
Comparative Risk Aversion

$\forall \tilde{x}$ if $E[V_1(\omega+\tilde{x})] - V_1(\omega) \leq 0$, then $E[V_2(\omega+\tilde{x})] - V_2(\omega) \leq 0$



- diffident around ω
- more diffident around x





$\forall \tilde{x}$ $\quad E\left[V_2(\omega+\tilde{x}) - V_2(\omega)\right] \leq m\, E\left[V_1(\omega+\tilde{x}) - V_1(\omega)\right]$

$\Longleftrightarrow V_2(\omega+x) - V_2(\omega) \leq m\left[V_1(\omega+x) - V_1(\omega)\right]$

$V_2'(\omega+x)\big|_{x=0} = m\, V_1'(\omega+x)\big|_{x=0}$ $\qquad m = V_2'(\omega)/V_1'(\omega)$

$V_2''(\omega+x)\big|_{x=0} \leq \underset{\uparrow\ V_2'(\omega)/V_1'(\omega)}{m} V_1''(\omega+x)\big|_{x=0}$ $\quad$ *local necessary condition*

$$V_2''(\omega) \leq \frac{V_2'(\omega)}{V_1'(\omega)} V_1''(\omega)$$

Central NSC $\Rightarrow V_2(\omega+x) - V_2(\omega) \leq \dfrac{V_2'(\omega)}{V_1'(\omega)}\left[V_1(\omega+x) - V_1(\omega)\right]$

$$\frac{V_2(\omega+x) - V_2(\omega)}{V_2'(\omega)} \leq \frac{V_1(\omega+x) - V_1(\omega)}{V_1'(\omega)} \Rightarrow \text{local NC}$$

Does $V_2''(\omega) \leq \dfrac{V_2'(\omega)}{V_1'(\omega)} V_1''(\omega)$ $\forall \omega \Rightarrow \dfrac{V_2(\omega+x) - V_2(\omega)}{V_2'(\omega)} \leq \dfrac{V_1(\omega+x) - V_1(\omega)}{V_1'(\omega)}$

$\forall \omega, x$?

$$\frac{V_2''(\omega)}{V_2'(\omega)} \leq \frac{V_1''(\omega)}{V_1'(\omega)}$$

$\frac{d}{d\omega}\log(V_2'(\omega)) \leq \frac{d}{d\omega}\log(V_1'(\omega))$

$\Rightarrow \log\left(\dfrac{V_2'(\omega)}{V_1'(\omega)}\right)$ is decreasing $\Longleftrightarrow \dfrac{V_2'(\omega)}{V_1'(\omega)}$ is decreasing

We know V_2'/V_1' is decreasing

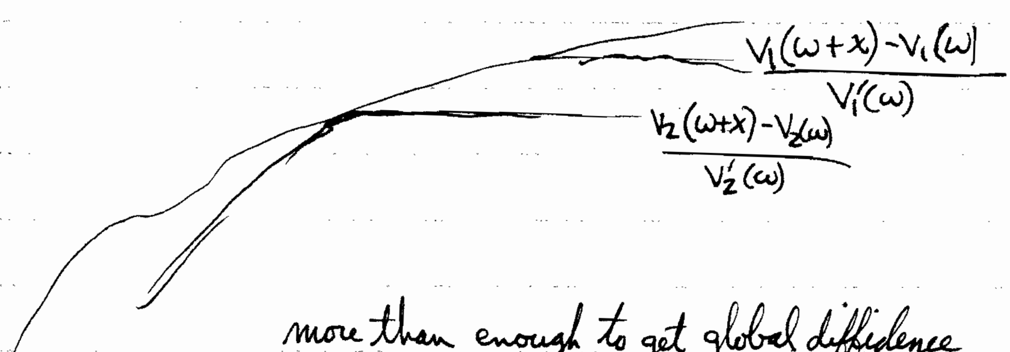
To prove:

$$\frac{\int_0^x V_2'(\omega+\xi)\,d\xi}{V_2'(\omega)} \leq \frac{\int_0^x V_1(\omega+\xi)\,d\xi}{V_1'(\omega)}$$

$$\int_0^x \left[\frac{V_2'(\omega+\xi)}{V_2'(\omega)} - \frac{V_1'(\omega+\xi)}{V_1'(\omega)}\right] d\xi \leq 0$$

$$\Leftarrow \frac{V_2'(\omega+\xi)}{V_2'(\omega)} - \frac{V_1'(\omega+\xi)}{V_1'(\omega)} \begin{cases} \leq 0 & \text{if } \xi \geq 0 \\ \geq 0 & \text{if } \xi \leq 0 \end{cases}$$

$$\frac{V_2'(\omega+\xi)}{V_2'(\omega)} \leq \frac{V_1(\omega+\xi)}{V_1'(\omega)} \quad \Leftrightarrow \quad \frac{V_2'(\omega+\xi)}{V_1'(\omega+\xi)} \begin{matrix} \leq \\ \geq \end{matrix} \frac{V_2'(\omega)}{V_1'(\omega)} \quad \begin{matrix} \text{if } \xi \geq 0 \\ \text{if } \xi \leq 0 \end{matrix}$$

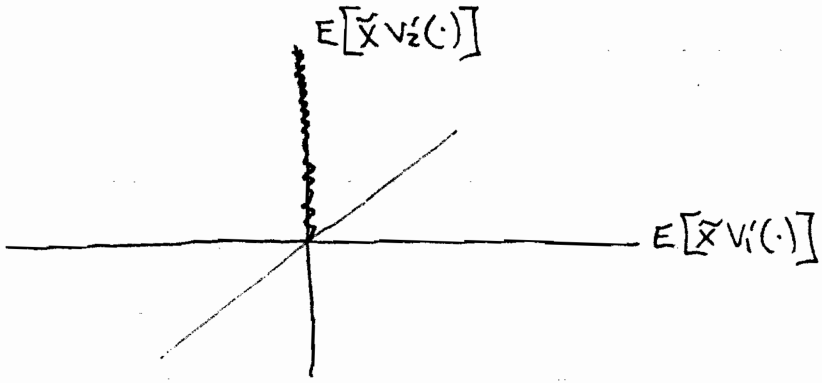


more than enough to get global diffidence

- 1 person wanting less of a risky asset than another person
  if $E[\tilde{x} V_1'(\omega+\tilde{x})] \leq 0$, then $E[\tilde{x} V_2'(\omega+\tilde{x})] \leq 0$
  Centrally greater risk aversion ($\Rightarrow$ centrally greater diffidence)
  (1) Do procedure we've been doing
  (2) Show it implies centrally greater diffedence
  (3) Show centrally greater diffedene $\not\Rightarrow$ centrally greater risk aversion

1.25 Check out Journal of Answers to Kimball

if $E[\tilde{x} V_1'(\omega+\tilde{x})]=0$, then $E[\tilde{x} V_2'(\omega+\tilde{x})] \leq 0$



$\exists m$

$$x V_2'(\omega+x) \leq m\, x V_1'(\omega+x)$$

$$V_2'(\omega+x) + x V_2''(\omega+x) \qquad m\left(V_1'(\omega+x) + x V_1''(\omega+x)\right)$$

$$2V_2''(\omega+x) + x V_2'''(\omega+x) \qquad m\left(2V_1''(\omega+x) + x V_1'''(\omega+x)\right)$$

let $x=0 \Rightarrow V_2'(\omega) = m V_1'(\omega) \quad \Rightarrow m = V_2'(\omega)/V_1'(\omega)$

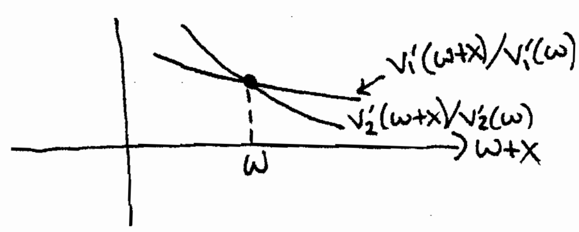
$$2V_2''(\omega) \leq m\, 2V_1''(\omega)$$

$$V_2''(\omega) \leq \left(V_2'(\omega)/V_1'(\omega)\right) V_1''(\omega) \quad \text{local NC}$$

$$\Downarrow ?$$

$$x V_2'(\omega+x) \leq \left(V_2'(\omega)/V_1'(\omega)\right) x V_1'(\omega+x) \quad \text{Central NSC}$$

$$x \frac{V_2'(\omega+x)}{V_2'(\omega)} \leq x \frac{V_1'(\omega+x)}{V_1'(\omega)}$$



$$\frac{V_2''(\omega)}{V_2'(\omega)} \leq \frac{V_1''(\omega)}{V_1'(\omega)} \iff \frac{\partial}{\partial\omega}\log(V_2'(\omega)) \leq \frac{\partial}{\partial\omega}\log(V_1'(\omega))$$

~~Proof~~ Notation $E[u(\tilde{x})] = P\cdot u$

~~Defn~~ $\int_a^b u(x)dF(x) = [u(x)F(x)]_a^b - \int_a^b u'(x)F(x)dx$

$= u(b) - \int_a^b u'(x)F(x)dx \qquad G(x) = \int_a^x F(\xi)\,d\xi$

$= u(b) - [u'(x)G(x)]_a^b + \int_a^b u''(x)G(x)dx$

$= u(b) - u'(b) + \int_a^b u''(x)G(x)dx$

$\mathbb{F} = F_2 - F_1$

$\mathbb{G} = G_2 - G_1$ difference in two random variables $-\int_a^b u'(x)\,\mathbb{F}(x)\,dx = -u'(b)\,\mathbb{G}(b) + \int u''(x)\,\mathbb{G}(x)dx$

$\mathbb{G}(b) = -\left(E[\tilde{x}_2] - E[\tilde{x}_1]\right)$

Polars: Def. $\mathcal{A}^\circ = \{B \mid B\cdot A \geq 0, \forall A \in \mathcal{A}\} = \mathcal{B}$ $\quad \mathcal{B}$ is polar of $\mathcal{A}$.

Suppose $P_2 \succsim P_1 \Rightarrow$ EU theory says $P_2 \succsim g P_2 + (1-g) P_1$

Theorem 1: If $\mathcal{A}$ and $\mathcal{B}$ are both closed convex cone, then $\mathcal{A} \cap \mathcal{B}$ is a closed convex cone (ccc)

Theorem 2: $ccc(\mathcal{A} \cup \mathcal{B}) = ccc(\mathcal{A}) + ccc(\mathcal{B})$

Theorem 3: $\mathcal{A}^\circ$ is a ccc.

Theorem 4: $(\mathcal{A}^\circ)^\circ = ccc(\mathcal{A})$

Theorem 5: $\mathcal{A}^\circ = (ccc\,\mathcal{A})^\circ$

Theorem 6: $(\mathcal{A} \cup \mathcal{B})^\circ = \mathcal{A}^\circ \cap \mathcal{B}^\circ$

Theorem 7: $(\mathcal{A} \cap \mathcal{B})^\circ = \mathcal{A}^\circ + \mathcal{B}^\circ$ (positive linear combination)

Say you have value functions V_1, V_2

$\mathcal{V} = \{V \text{ s.t. } V \text{ more risk averse than } V_1,\ V \text{ less risk averse than } V_2\}$

$\mathcal{A} = \{V \mid V \text{ m.r.a. } V_1\}$

$\mathcal{B} = \{V \mid V \text{ l.r.a. } V_2\}$

Then random variables such that these are true : $(\mathcal{A} \cap \mathcal{B})^\circ = \mathcal{A}^\circ + \mathcal{B}^\circ$

~~Theorem 18:~~

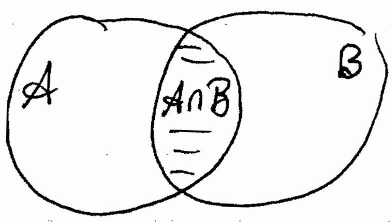
convex cones — closed

Proof of 1: $\sum_i \alpha_i A_i \in \mathcal{A}$ $\forall \alpha_i \geq 0$ $\quad \sum_i \beta_i B_i \in \mathcal{B}$ $\forall \beta_i \geq 0$ $\quad , A_i \in \mathcal{A}\ \forall i \Rightarrow \lim_{i\to\infty} A_i \in \mathcal{A}$

$B_i \in \mathcal{B}\ \forall i \Rightarrow \lim_{i\to\infty} B_i \in \mathcal{B}$

Let $C_i \in \mathcal{A} \cap \mathcal{B} \Rightarrow \sum_i \gamma_i C_i \in \mathcal{A}$ and $\sum_i \gamma_i C_i \in \mathcal{B} \Rightarrow \sum_i \gamma_i C_i \in \mathcal{A} \cap \mathcal{B}$

$\lim_{i\to\infty} C_i \in \mathcal{A}$, $\lim_{i\to\infty} C_i \in \mathcal{B} \Rightarrow \lim_{i\to\infty} C_i \in \mathcal{A} \cap \mathcal{B}$



Proof of 2: $\sum_i \alpha_i A_i \in ccc(\mathcal{A})$, $\sum_i \beta_i B_i \in ccc(\mathcal{B})$ $\quad \alpha_i \geq 0$, $\beta_i \geq 0$

$\lim_{i\to\infty} A_i \in ccc(\mathcal{A})$, $\lim_{i\to\infty} B_i \in ccc(\mathcal{B})$

$C_i \in \mathcal{A} \cup \mathcal{B}$

$$\sum_i \gamma_i C_i = \underbrace{\sum_i \alpha_i A_i}_{\in\, ccc(\mathcal{A})} + \underbrace{\sum_i \beta_i B_i}_{\in\, ccc(\mathcal{B})}, \quad \gamma_i \geq 0$$

Proof of 3: $\mathcal{B} = \mathcal{A}^\circ = \{B \mid B \cdot A \geq 0, \forall A \in \mathcal{A}\}$

if $B_i \in \mathcal{A}^\circ$, then $\sum_i \beta_i B_i \in \mathcal{A}^\circ$ (want to show)

$B_i \cdot A \geq 0 \ \forall A \in \mathcal{A}\ \forall i$

$\sum_i \beta_i B_i \cdot A = \sum_i \beta_i (B_i \cdot A) \geq 0 \ \forall A \in \mathcal{A}$ $\quad$ convex cone part

$B_i \cdot A \geq 0 \quad \forall A \in \mathcal{A} \quad \forall i$

$\left(\lim_{B_i \to \infty} B_i\right) \cdot A = \lim_{i \to \infty} (B_i \cdot A) \geq 0 \quad \forall A \in \mathcal{A}$ closed

~~Proof of 4:~~

Q: ~~what is the set~~ $\forall \omega$ if $E[\tilde{x}] = 0 \Rightarrow E[V(\omega + \tilde{x})] \leq V(\omega)$

Polar $\Updownarrow$ $V''(\omega) \leq 0$

Polar of $(V''(x) \leq 0)$ is the set of mean-preserving spreads

* Making ~~a~~ the set larger $\Rightarrow$ make the polar smaller

1.30 The polar of increasing functions is ... 1st order stochastic dominant risks

(4) For any $\mathcal{A} \subset \mathbb{R}^n$, $(\mathcal{A}^\circ)^\circ = ccc(\mathcal{A})$

(a) $x \in ccc(\mathcal{A}) \Rightarrow x \in (\mathcal{A}^\circ)^\circ$

$ccc(\mathcal{A}) \subset (\mathcal{A}^\circ)^\circ$

$\mathcal{B} = \mathcal{A}^\circ = \{B \mid B \cdot A \geq 0, \forall A \in \mathcal{A}\}$

$\forall B \in \mathcal{B}, \quad B \cdot \sum_i \alpha_i A_i = \sum_i \alpha_i (B \cdot A_i) \geq 0$

if $B \cdot A_i \geq 0 \quad \forall A_i \in \mathcal{A}$, then $\lim_{i \to \infty} B \cdot A_i \geq 0$

$\Rightarrow ccc(\mathcal{A}) \subset \mathcal{B}^\circ$

$ccc(\mathcal{A}) \subset (\mathcal{A}^\circ)^\circ$

(b) $x \notin ccc(\mathcal{A}) \Rightarrow x \notin (\mathcal{A}^\circ)^\circ$

(i) First, if $\mathcal{A}$ is in a subspace S of $\mathbb{R}^n$, then so is $(\mathcal{A}^\circ)^\circ$

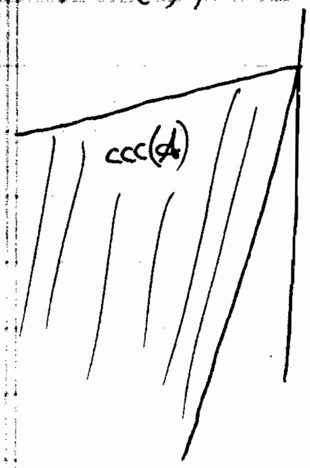
Proof: $\forall y \in \perp S$, $y \cdot A = 0$ for all $A \in \mathcal{A}$.

So, $\perp S \subset \mathcal{A}^\circ$.

If $x \notin S$, then $x \cdot y \neq 0$ for some $y \in \perp S$.

Also, $x \cdot (-y) \neq 0$. Therefore, $x \notin (\mathcal{A}^\circ)^\circ$

(b) (ii) if $x \in S$, where S is the minimal subspace including $\mathcal{A}$, but $x \notin ccc(\mathcal{A})$, then $x \notin (\mathcal{A}^\circ)^\circ$.



Separating hyperplane.

$(\mathcal{A}^\circ)^\circ \subset ccc(\mathcal{A})$

(5) $\mathcal{A}^\circ = (ccc(\mathcal{A}))^\circ$

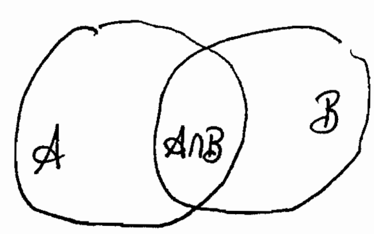
Proof: $((\mathcal{A}^\circ)^\circ)^\circ = (ccc(\mathcal{A}))^\circ = ccc(\mathcal{A}^\circ) = \mathcal{A}^\circ$

(6) $(\mathcal{A} \cup \mathcal{B})^\circ = \mathcal{A}^\circ \cap \mathcal{B}^\circ$

$\{x \mid x \cdot A \geq 0\ \forall A \in \mathcal{A} \text{ and } x \cdot B \geq 0\ \forall B \in \mathcal{B}\} = (\mathcal{A} \cup \mathcal{B})^\circ$

$\overset{?}{=} \{x \mid x \cdot A \geq 0\ \forall A \in \mathcal{A}\} \cap \{x \mid x \cdot B \geq 0\ \forall B \in \mathcal{B}\} = \mathcal{A}^\circ \cap \mathcal{B}^\circ$

(7) $(\mathcal{A} \cap \mathcal{B})^\circ = \mathcal{A}^\circ + \mathcal{B}^\circ$ if $\mathcal{A}$ and $\mathcal{B}$ are ccc



(a) $\mathcal{A}^\circ + \mathcal{B}^\circ \subset (\mathcal{A} \cap \mathcal{B})^\circ$

$x \in \mathcal{A}^\circ$, $y \in \mathcal{B}^\circ$

$x \cdot A \geq 0\ \forall A \in \mathcal{A}$ $\quad$ $y \cdot B \geq 0\ \forall B \in \mathcal{B}$

$x \cdot C \geq 0\ \forall C \in (\mathcal{A} \cap \mathcal{B})$

$y \cdot C \geq 0\ \forall C \in (\mathcal{A} \cap \mathcal{B})$

$\Rightarrow (x+y) \cdot C \geq 0$

(b) $(\mathcal{A} \cap \mathcal{B})^\circ \subset \mathcal{A}^\circ + \mathcal{B}^\circ$

since $\mathcal{A}$ & $\mathcal{B}$ are ccc

$$(\mathcal{A} \cap \mathcal{B})^\circ = \left[(\mathcal{A}^\circ)^\circ \cap (\mathcal{B}^\circ)^\circ\right]^\circ = \left[(\mathcal{A}^\circ \cup \mathcal{B}^\circ)^\circ\right]^\circ = \text{ccc}(\mathcal{A}^\circ \cup \mathcal{B}^\circ) = \mathcal{A}^\circ + \mathcal{B}^\circ$$

Interesting sets

(Polar: 5. $V' \geq 0$) basis

(1) change in r.v.'s from ω to $\omega + \tilde{x}$ (basis: $\omega \to \omega + x$)

(2) first order stochastic dominance changes (basis: $\omega \to \omega - \theta$, $\theta \geq 0$)

(3) ~~second~~ mean preserving spreads (basis $\omega \to \begin{cases} \omega - (1-p)\theta & \text{w/ prob } p \\ \omega + p\theta & 1-p \end{cases}$)

(Polar: 6. $V'' \leq 0$) basis

if $E[\tilde{x}] = 0$, then $E[V(\omega + \tilde{x})] - V(\omega) \leq 0$

(4) SSD changes (basis. $\omega \to \omega - \theta$; $\omega \to \begin{cases} \omega - (1-p)\theta & \text{w/prob } p \\ \omega + p\theta & 1-p \end{cases}$) | Polar 7. $V' \geq 0$, $V'' \leq 0$ basis

(5) ~~V~~

Polars

8. V mra than $\bar{V}$

$$V \text{ s.t. } V(x) = \phi(\bar{V}(x)) \text{ for some increasing concave } \phi$$
$$= \alpha_0 + \sum_i \alpha_i \min\{0, V(x) - \bar{V}(\theta)\}$$

9. V lra than $\bar{V}$

$$V \text{ s.t. } V(x) = \phi(\bar{V}(x)) \text{ for some increasing convex } \phi$$
$$= \alpha_0 + \sum_i \alpha_i \max\{0, V(x) - \bar{V}(\theta)\}$$

intersection is not a nice set. The polar ~~[illegible]~~ stuff will help here.

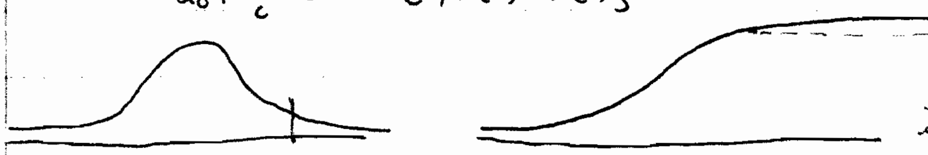
10. F is worse by MLR order than $\bar{F}$.

$F'(x)/\bar{F}'(x)$ is decreasing

(a lot like u_2'/u_1' is decreasing)

$F = \phi(\bar{F})$, ϕ concave

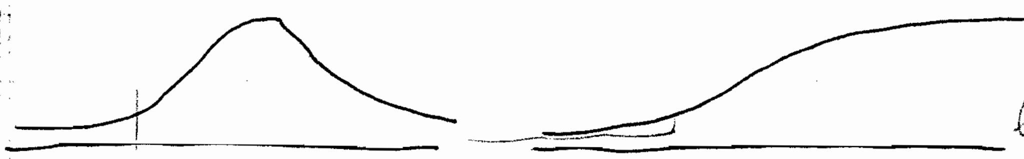
$= \alpha_0 + \sum_i \alpha_i \min\{0, F(x) - \bar{F}(\theta)\}$



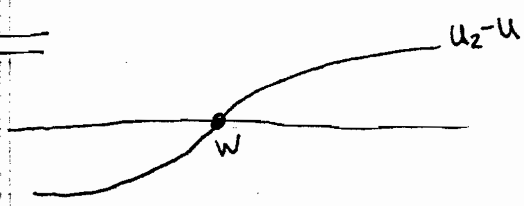
chopping off top part of dist and adding it to the rest randomly

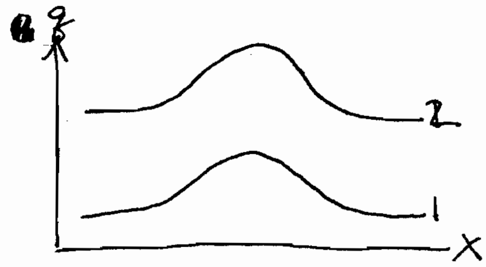
11. F is better by MLR order than $\bar{F}$

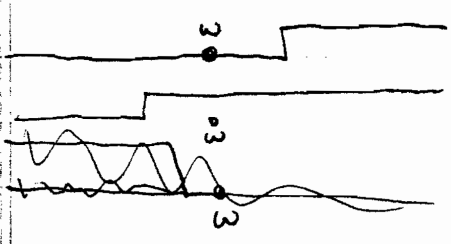
$F = \alpha_0 + \sum_i \alpha_i \max\{0, F(x) - \bar{F}(\theta)\}$

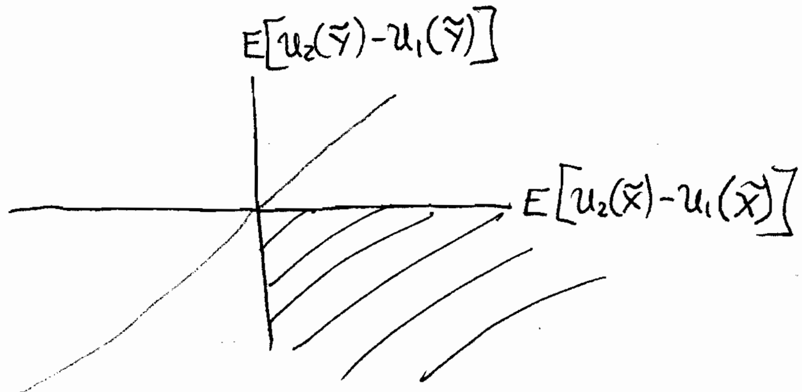


chopping off bottom part









Have $u_2(\omega) = u_1(\omega)$

$E[u_2(\tilde{y}) - u_1(\tilde{y})] \geq m\left(E[u_2(\tilde{x}) - u_1(\tilde{x})]\right)$

$\exists m$ s.t. $1 - F_y(\theta) \geq m(1 - F_x(\theta)) \quad \forall \theta \geq \omega$

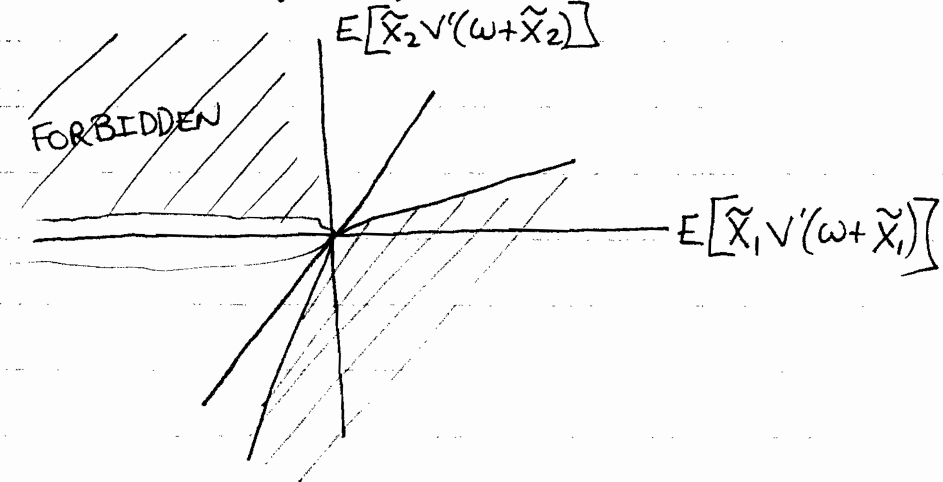
$-F_y(\theta) \geq m(-F_x(\theta)) \quad \forall \theta < \omega \iff F_y(\theta) \leq m F_x(\theta) \quad \forall \theta < \omega$

2.1 Example relevant to #6 on exam $\quad \forall\, \tilde{X}_2 \overset{FSD}{\lesssim} \tilde{X}_1$

Assume $V' \geq 0$. If $E[\tilde{X}_1 V_1'(\omega+\tilde{X}_1)] \leq 0$

$(\Leftarrow)$ then $E[\tilde{X}_2 V'(\omega+\tilde{X}_2)] \leq 0$

Find shapes of value functions where this is true.



$\tilde{X}_2 \overset{FSD}{\leq} \tilde{X}_1, \quad \tilde{Y}_2 \overset{FSD}{\leq} \tilde{Y}_1 \Rightarrow \alpha \tilde{X}_2 + (1-\alpha)\tilde{X}_2 \overset{FSD}{\lesssim} \alpha \tilde{X}_1 + (1-\alpha)\tilde{Y}_1$

F for X, G for Y

$F_2(x) \geq F_1(x) \qquad G_2(x) \geq G_1(x_1)$

$F_2(x) - F_1(x) \geq 0 \qquad G_2(x) - G_1(x) \geq 0$

$\alpha(F_2 - F_1) + (1-\alpha)(G_2 - G_1) \geq 0$

for all x and $\in$.

$\exists m : E[\tilde{X}_2 V'(\omega+\tilde{X}_2)] \leq m E[\tilde{X}_1 V'(\omega+\tilde{X}_1)] \quad \forall (\tilde{X}_1, \tilde{X}_2)$ s.t. $\tilde{X}_2 \overset{FSD}{\leq} \tilde{X}_1$.

- Taking prob. mass from lower point to higher $(\overset{\forall}{x} \to x+\theta \quad \overset{\forall}{\theta} \geq 0)$ basis for $\tilde{X}_2 \to \tilde{X}_1$

$xV'(\omega+x) \leq m(x+\theta)V'(\omega+x+\theta) \quad \forall x, \forall \theta \geq 0$

$\forall \theta \quad xV'(\omega+x) - m(x+\theta)V'(\omega+x+\theta) \leq 0$

In particular, $\theta = 0 \Rightarrow (1-m)xV'(\omega+x) \leq 0$

For $V' > 0$, $m=1$. If $V'(\omega) = 0$, ? $\qquad V'(\omega+x) \in (V'(\omega) - \varepsilon, V'(\omega)+\varepsilon)$

$x V'(\omega+x) \leq (x+\theta) V'(\omega+x+\theta)$

$f(\omega,x) = x V'(\omega+x)$

$f(\omega,x+\theta) \geq f(\omega,x)$ $\vee$ increasing in x

$\frac{\partial}{\partial x} x V'(\omega+x) \geq 0 \quad \forall x$

$V'(\omega+x) + x V''(\omega+x) \geq 0$

$1 + \frac{x V''(\omega+x)}{V'(\omega+x)} \geq 0 \qquad 1 \geq -\frac{x V''(\omega+x)}{V'(\omega+x)}$

If $x > 0$, then $\frac{-V''(\omega+x)}{V'(\omega+x)} \leq \frac{1}{x}$

≤ 0, then $\frac{-V''(\omega+x)}{V'(\omega+x)} \geq \frac{1}{x}$

neg. #

Suppose this is true $\forall \omega$. $\forall \omega$, $\forall x > 0$, $\frac{-V''(\omega+x)(\omega+x)}{V'(\omega+x)} \leq \frac{\omega}{x} + 1$

limit $\omega + x \geq 0$

FSD isn't enough for people to want less of risky asset. Need ~~[illegible]~~ implausibly low coefficient of risk aversion.

Multiperiod models

- Recursive approach
  - Recursive definition of utility

$\nu_t \leftarrow$ lifetime utility   unmaximized counterpart to value function V

$$\nu_t = \Psi(K_t, X_t, E_t[\nu_{t+h}]; h) \qquad h = \Delta t$$

K_t = vector of state variables (parameters, ~~exogenous~~ endogenous state variables, time)

X_t = vector of control variables

terminal condition $\nu_{T+h} \equiv 0$

Examples:

(1) $\nu_t = h\,u^t(K_t, X_t) + e^{-\rho h} E_t[\nu_{t+h}]$

$$\Updownarrow$$

$$\nu_t = h \sum_{n=0}^{\frac{T-t}{h}} e^{-\rho n h}\, u^{t+nh}(K_{t+nh}, X_{t+nh})$$

(2) $\nu_t = h\,u^t(K_t, X_t) + e^{-h\rho(E_t[\nu_{t+h}])} E_t[\nu_{t+h}]$

$$\frac{\nu_t - e^{-h\rho(\nu_{t+h})}\nu_{t+h}}{h} = u^t(K_t, X_t)$$

$$\frac{\nu_{t-h} - e^{-h\rho(\nu_t)}\nu_t}{h} = u^{t-h}(K_{t-h}, X_{t-h})$$

$$-\dot{\nu}_{t-h} - \rho \exp\left(h - \rho(E_{t-h}(\nu_t))\right) E_{t-h}[\nu_t] = u^{t-h}(K_{t-h}, X_{t-h})$$

$h \to 0$ $$-\dot{\nu}_{t^-} - \rho\, E_{t^-}(\nu_t) = u^t(K_t, X_t)$$

look at old notes (#30-36, stochastic calculus)

3. Kreps-Porteus Preferences — separate intertemporal substitution and risk aversion

$$v_t = hU^t(K_t, X_t) + \exp(-\rho h)\,\Phi^{-1}\left(E_t\left[\Phi(v_{t+h})\right]\right)$$

$$\Leftrightarrow \Phi(v_t) = \Phi\left(hU^t(K_t, X_t) + \exp(-\rho h)\,\Phi^{-1}\left(E\left[\Phi(v_{t+h})\right]\right)\right)$$

$$U_t = \Phi\left(hU^t(K_t, X_t) + \exp(-\rho h)\,\Phi^{-1}\left(E\left[U_{t+h}\right]\right)\right)$$

2.6 General formulation of lifetime utility

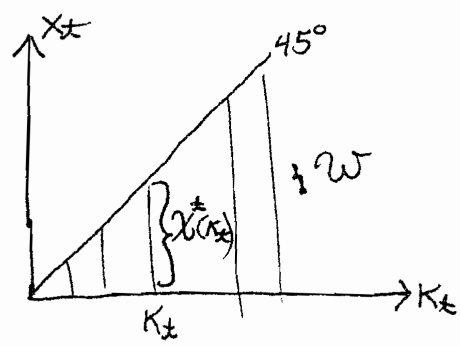
$$v_t = \varphi^t(K_t, X_t, E_t[v_{t+h}]; h)$$

$$v_{T+h} \equiv 0 \quad \text{for } h > 0$$

Constraints

- contemporaneous constraints : $X_t \in \mathcal{X}^t(K_t) \Leftrightarrow (K_t, X_t) \in \mathcal{W}$

e.g. $0 \le X_t \le K_t$



- intertemporal transition equation

$$K_{t+h} = \Gamma^t(K_t, X_t, \omega_{t+h}) \quad , \quad \omega_{t+h} = \text{vector of uniform } [0,1] \text{ random variable}$$

e.g. (1) $K_{t+h} = K_t + hA^t(K_t, X_t) \quad \Rightarrow \dot{K}_t = A(K_t, X_t)$

(2) $$K_{t+h} = \begin{cases} K_t + hA^t(K_t, X_t) + \sqrt{h\,\Omega^t(K_t, X_t)} & \text{w/ } p = 1/2 \\ K_t + hA^t(K_t, X_t) - \sqrt{h\,\Omega^t(K_t, X_t)} & \text{w/ } p = 1/2 \end{cases}$$

(3) same continuous time limit as (2)

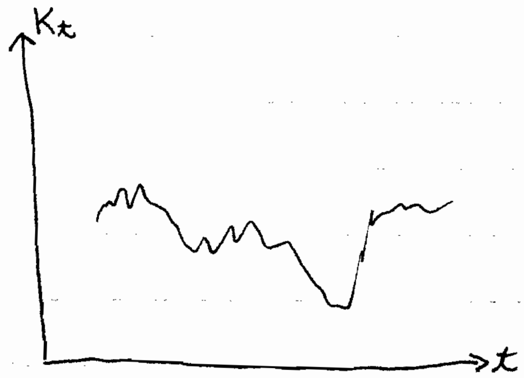
$$K_{t+h} = K_t + hA^t(K_t, X_t) + \sqrt{h\,\Omega^t(K_t, X_t)}\,\tilde{\varepsilon}_{t+h}, \quad E[\tilde{\varepsilon}_{t+h}] = 0, \; V[\tilde{\varepsilon}_{t+h}] = 1$$

$\tilde{\varepsilon}_{t+h}$ has finite support, $\tilde{\varepsilon}_{t+h}$ is iid

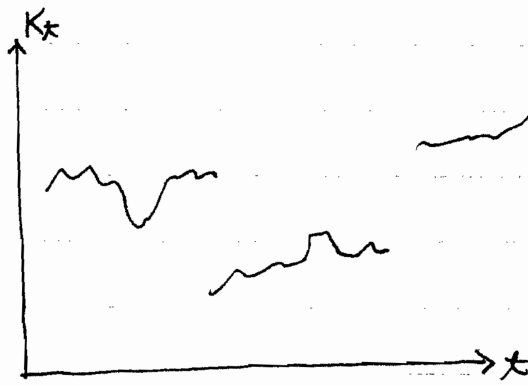
(4) $K_{t+h} = \begin{cases} K_t + hA^t(K_t, X_t) + \sqrt{h\,\Omega^t(K_t, X_t)}\,\tilde{\varepsilon}_{t+h} & \text{w/ PROB } 1-hp \\ \Phi(K_t, X_t, \tilde{e}_{t+h}) & \text{w/ PROB } hp \end{cases}$

small prob of ~~something~~ something big happening

Start w/ (2) or (3)

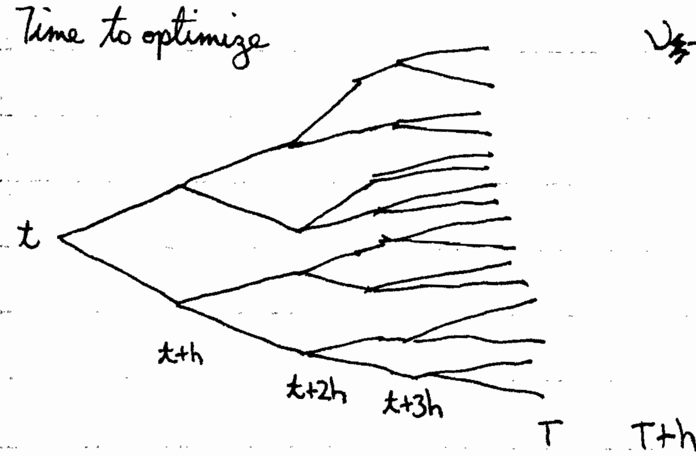


(4)



Time to optimize

$V_{T+h} \equiv 0$



Backwards recursion

Mathematical induction : prove for $n=1$, prove true $n \Rightarrow$ true for $n+1$ $\Rightarrow$ true $\forall n$

Recursion: mathematical induction for V_{T+h-nh}. Prove for $n=0$. Prove true for $n \Rightarrow$ true for $n+1$ $\Rightarrow$ true for all $n=0,1,2,\ldots$

V_t · Proove for $t=T+h$

· Prove that if true for $t+h$, then true for t $\Rightarrow$ true for all t.

V^t value function $V^t(K_t) = \max_{X_t \in \mathcal{X}(K_t)} \psi^t(K_t, X_t, E_t[V^{t+h}(K_{t+h})])$

This is the Bellman equation

dynamic consistency says time t's self $V^{t+h}(K_{t+h}) = $ time $t+h$'s self $V^{t+h}(K_{t+h})$.

Step 0: Prove $V^{T+h} \in P$ (has property P)

Step 1: Show that $V^{t+h}_{(K_{t+h})} \in P \Rightarrow (F^t, \mathcal{W}^t) \in Q$, where $F^t(K_t, X_t) = \psi^t(K_t, X_t, E_t[V^{t+h}(\Gamma^t(K_t, X_t, \tilde{\omega}_{t+h}))])$

$=$ Bellman maximand

remember, $X_t \in \mathcal{X}(K_t) \Leftrightarrow (K_t, X_t) \in \mathcal{W}^t$

Step 2: Show that $(F^t, \mathcal{W}^t) \in Q \Rightarrow V^t \in P$

Step 3: By mathematical induction (recursion), $V^t \in P$ $\forall t = T+h-nh$.

Easy optional steps (can do in either order; can done none, one, or both)

Step 4: $T \to \infty$ show $V^t(K_t; T, h) \in P$ $\forall T \Rightarrow \lim_{T\to\infty} V^t(K_t; T, h) \in P$.

Typically, $V^t(K_t; T, h) \in P \Leftrightarrow \Phi(V^t) \geq 0$ $\longrightarrow T$

example: V^t increasing in K_t. $\Phi(V) = V^t(K_t+\delta) - V^t(K_t) \geq 0 \quad \forall K_t$

functional is a function of a function (sort of)

Step 5: Show that $V^t(K_t;T,h) \in P \ \forall h \Rightarrow \lim_{h \to 0} V^t(K_t;T,h) \in P$

Example $\forall K_t' \geq K_t$, $V^t(K_t') \geq V^t(K_t) \Leftrightarrow V^t(K_t') - V^t(K_t) \geq 0$

$\Leftrightarrow \quad \forall \delta \geq 0 \quad V^t(K_t+\delta) - V^t(K_t) \geq 0 \quad, \forall K_t$

Want to show this ($V(\cdot)$ increasing in K_t)

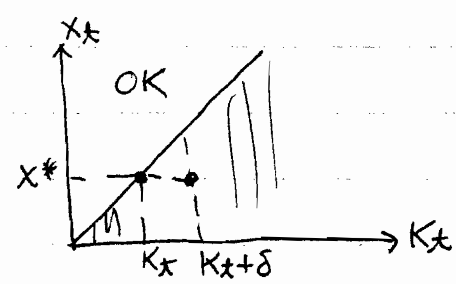
$$V^t(K_t+\delta) - V^t(K_t) = \max_{(K_t^{+\delta}, X_t) \in \mathcal{W}^t} \psi^t\left(K_t^{+\delta}, X_t, E_t\left[V^{t+h}\left(\Pi^t(K_t^{+\delta}, X_t; \omega_{t+h})\right)\right]\right)$$
$$- \max_{\{K_t, X_t\} \in \mathcal{W}^t} \psi^t\left(K_t, X_t, E_t\left[V^{t+h}\left(\Pi^t(K_t, X_t; \omega_{t+h})\right)\right]\right)$$

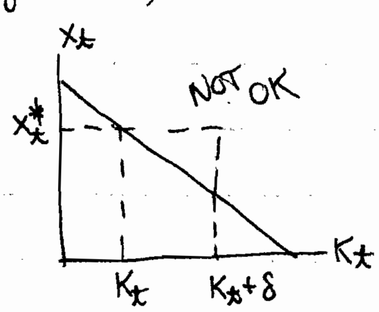
Let $X^* = \arg\max \psi^t\left(K_t, X_t, E_t\left[V^{t+h}\left(\Pi^t(K_t, X_t; \omega_{t+h})\right)\right]\right)$

$$\overset{?}{\geq} \psi^t\left(K_t+\delta, X_t^*, E_t\left[V^{t+h}\left(\Pi^t(K_t+\delta, X_t^*; \omega_{t+h})\right)\right]\right) - \psi^t\left(K_t, X_t^*, E_t\left[V^{t+h}\left(\Pi^t(K_t, X_t^*; \omega_{t+h})\right)\right]\right)$$

$F^t(K_t+\delta, X_t^*) - F^t(K_t, X_t^*)$ want to show this ≥ 0 (need F increasing in K)

if $(K_t, X_t) \in \mathcal{W}^t \Rightarrow (K_t+\delta, X_t) \in \mathcal{W}^t$



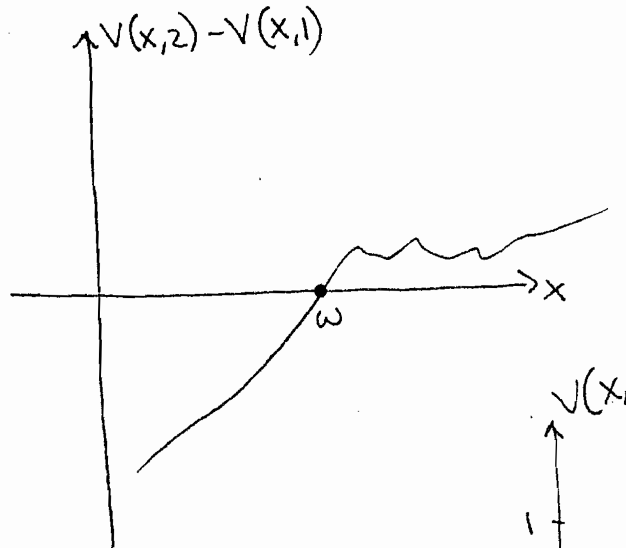


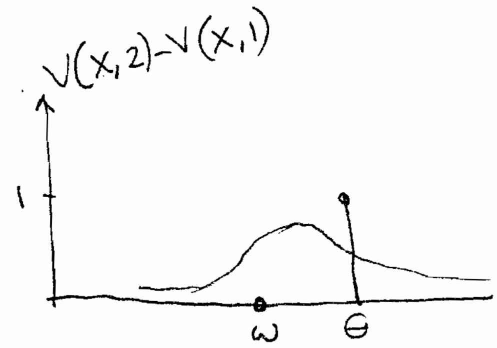
2.8 Referee reports

(1) (a) Choose $\alpha = 2$ if face $\tilde{X}_1 \Rightarrow$ choose $\alpha = 2$ if face $\tilde{X}_2$.

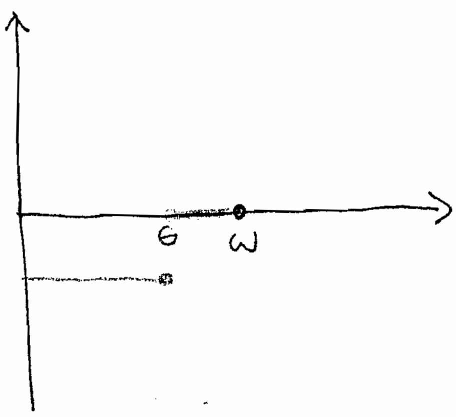
single crossing property

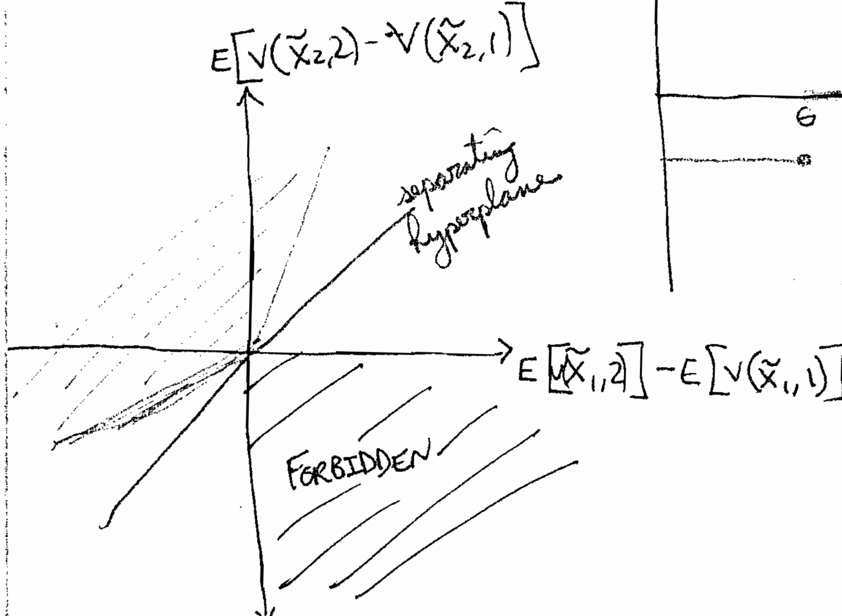
(b)





Find basis:





$\exists m \geq 0$ $\quad E\left[V(\tilde{X}_2, 2) - V(\tilde{X}_2, 1)\right] \geq m\, E\left[V(\tilde{X}_1, 2) - V(\tilde{X}_1, 1)\right]$

if this is true for any value function in the set (*), it must be true for the basis functions.

$\theta > \omega$: $E\left[b_\theta(\tilde{x}_2)\right] \geq m E\left[b_\theta(\tilde{x}_1)\right]$

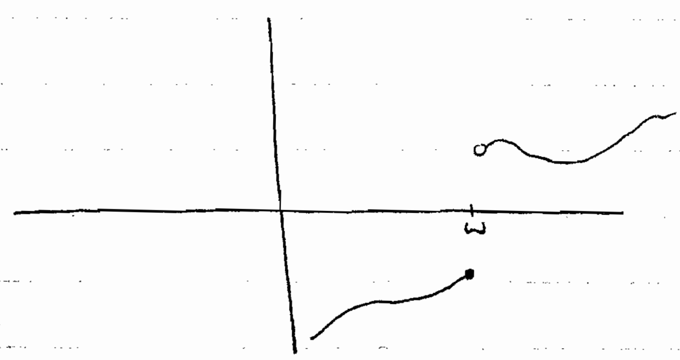
$\exists m \geq 0$

$\int_a^b f(x) b_\theta(x) dx = f(\theta)$ (1) $f_2(\theta) \geq m f_1(\theta)$ $\sum_a^b$

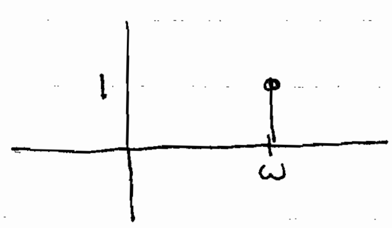
(2) $-F_2(\theta) \geq -m F_1(\theta)$

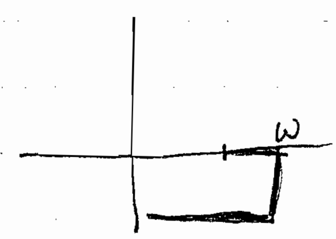
$\theta < \omega$ $\int_a^b f(x) b_\theta(x) dx = \int_a^\theta (-f(x)) dx = -F(\theta)$ $\sum_a^\theta$

(c)



add to set of basis functions





(1) $f_2(\theta) \geq m f_1(\theta)$, $\theta \geq \omega$

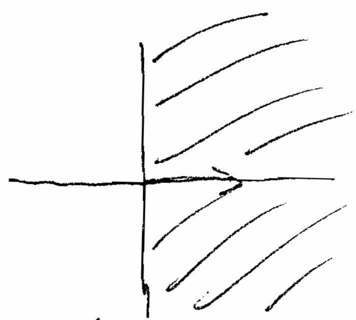
(2) $-F_2(\theta) \geq -m F(\theta)$, $\theta \leq \omega$

(2) $(\mathbb{R}^2)^\circ = 0$ ~~ccc~~

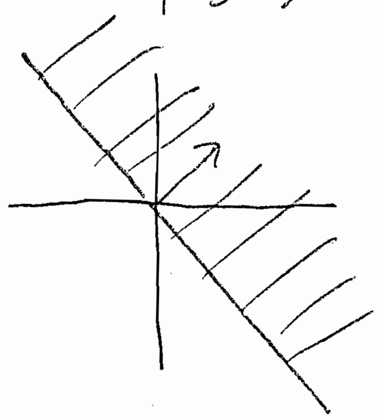
only point w/
non-negative dot product with any vector

$\{(0,0)\}^\circ = \mathbb{R}^2$

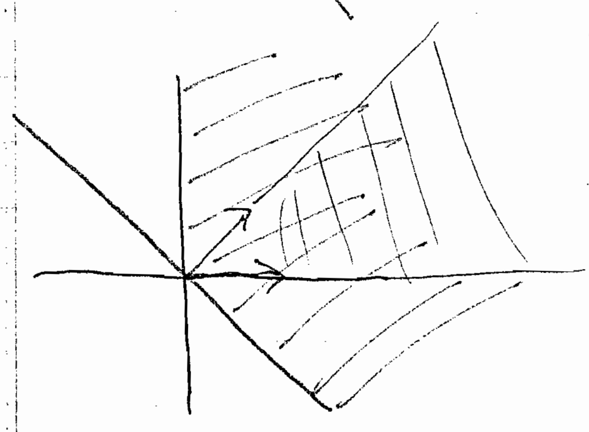
(0,0) is a ccc, as is $\mathbb{R}^2$



$\{(1,0)\}^\circ = \{(x,y) | : x \geq 0, \forall y\}$



say this is risk aversion, $\lesssim 6$ ⇒ changes in rV on purple line



Purple/Green polars ccc of each other

$(A \cap B)^\circ = A^\circ + B^\circ$
any changes in ~~rb~~ rV in purple cone

~~$A = \{V(x) | V(x)$~~

$$\int_a^b V(x)[f_2(x)\,\cancel{dx} - f_1(x)]\,dx = \overset{0}{V(x)[F_2(x) - F_1(x)]\Big|_a^b} - \int_a^b V'(x)(F_2(x) - F_1(x))\,dx$$

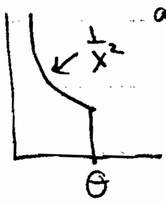
$\phi\left(\frac{-1}{x}\right)$ where $\phi' \geq 0$, $\phi'' \leq 0$ (risk aversion $= 2$ for $\frac{-1}{x}$)

$\min\{0, V(x) - V(\theta)\}$ basis $b_\theta(x)$

marginal value $c_\theta(x) = b'_\theta(x) = \begin{cases} 0 & \text{if } x > \theta \\ V'(x) & \text{if } x < \theta \end{cases}$

$\boxed{2 \succsim 1}$

$$\int_a^b V(x)\left(f_2(x) - f_1(x)\right)dx = -\int_a^b b'_\theta(x)\left[F_2(x) - F_1(x)\right]dx$$



$$= -\int_a^\theta V'(x)\left(F_2(x) - F_1(x)\right)dx \overset{?}{\geq} 0 \quad \forall \theta \in [a,b]$$

$\bar{V}(x) = \frac{-1}{x}$

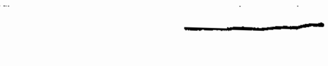
$\bar{V}'(x) = \frac{1}{x^2}$

$$\Rightarrow -\int_a^b \left(\frac{1}{x^2}\right)\left(F_2(x) - F_1(x)\right)dx \geq 0 \quad \forall \theta \in [a,b]$$

NSC for all value functions with rra ≥ 2 to prefer $\tilde{X}_2$ to $\tilde{X}_1$. $\mathcal{A}^0$

$\mathcal{B}$ = V w/ rel risk aversion ≤ 6

Basis of utility function

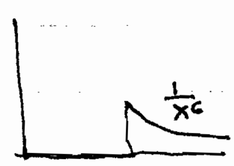


marginal utility basis $c_\theta(x) = \begin{cases} 0 & x < \theta \\ \frac{1}{x^6} & x > \theta \end{cases}$

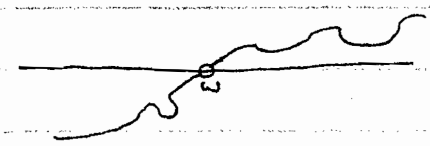
careful w/sign

$$\int_\theta^b \frac{1}{x^6}\left(F_1(x) - F_2(x)\right) \geq 0 \quad \forall \theta \in [a,b]$$

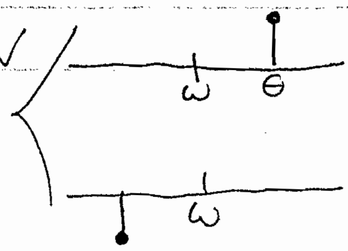
criterion for polar of B



(4) (a) $\theta\left[V(\omega+\theta)-V(\omega)\right]\geq 0$



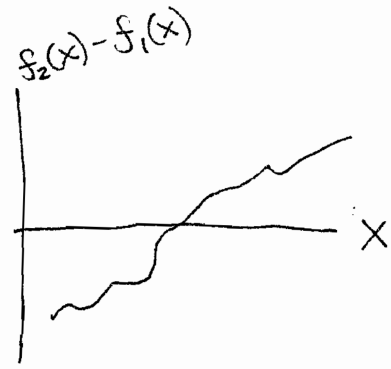
Basis for V



$\sum_{a}^{b}\left[f_2(x)-f_1(x)\right]b_\theta(x)\geq 0 \quad \forall b_\theta(x)$

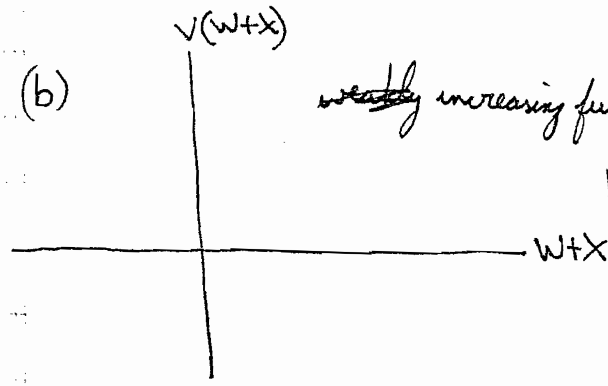
$f_2(\theta)-f_1(\theta)\geq 0 \quad \forall\theta\geq\omega$

$-(f_2(\theta)-f_1(\theta))\geq 0 \quad \forall\theta\leq\omega$



$\Rightarrow$ single crossing of densities

(b)



~~weakly~~ increasing functions $V(x)$

polar of this is first order stochastic dominance

FSD is ccc of densities that cross at different values.

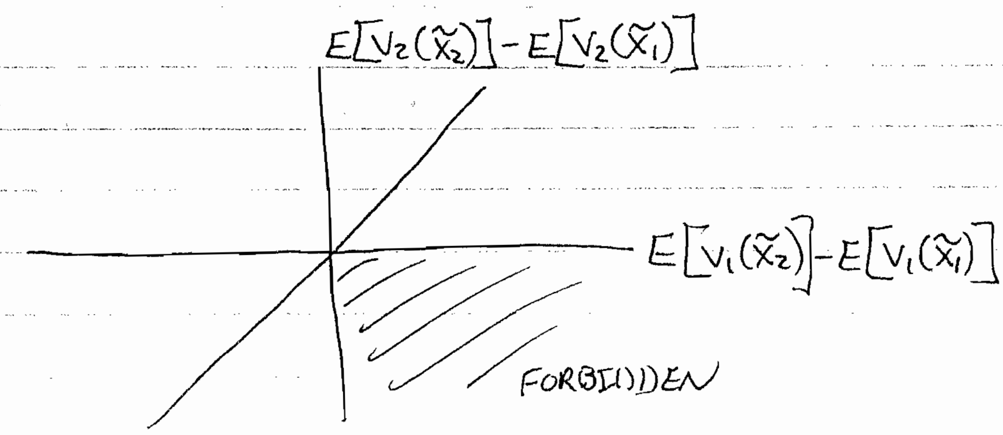
(5) $\exists m\geq 0$

$E\left[x^2V_2'''(\omega+x)\right]\leq m\,E\left[x^2V_1'''(\omega+x)\right]$

$x^2\,V_2'''(\omega+x)\leq m\,x^2\,V_1'''(\omega+x)$

$V_2'''(\omega+x)\leq m\,V_1'''(\omega+x)$ Draw picture of these

(6)



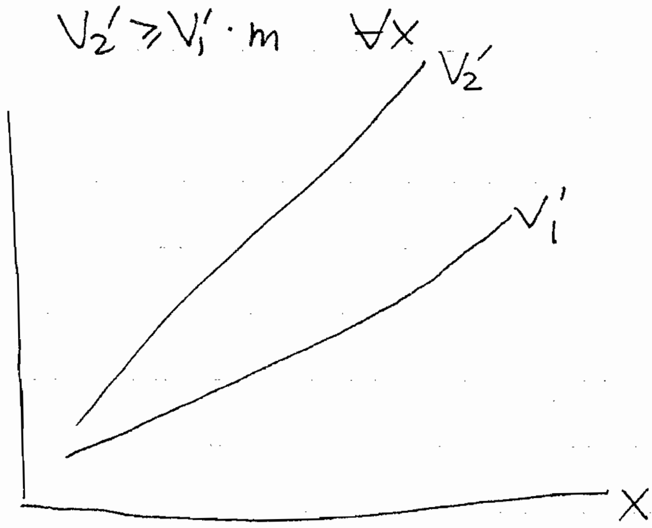
$$E[V_2(\tilde{X}_2)] - E[V_2(\tilde{X}_1)] \geq m\left(E[V_1(\tilde{X}_2)] - E[V_1(\tilde{X}_1)]\right.$$

$$\Updownarrow$$

$$V_2(x+\theta) - V_2(x) \geq m\left[V_1(x+\theta) - V_1(x)\right]$$

$$\lim_{\theta \to 0^+} \frac{V_2(x+\theta) - V_2(x)}{\theta} \geq \lim_{\theta \to 0^+} \frac{V_1(x+\theta) - V_1(x)}{\theta}\, m$$

$$V_2' \geq V_1' \cdot m \quad \forall x$$



2.13 $V^t(K_t) = \max_{0\le C_t\le K_t} U(C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_t - C_t) + \tilde{Y}_{t+1}\right)\right]$

$\mathcal{X}(K_t) = [0, K_t]$

$V^t(K_t+\theta) - V^t(K_t) = \max_{C_t\in[0,K_t+\theta]} U(C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_t+\theta-C_t)+\tilde{Y}_{t+1}\right)\right]$

$\forall\theta\ge 0 \qquad - \max_{C_t\in[0,K_t]} U(C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_t-C_t)+\tilde{Y}_{t+1}\right)\right]$

- want to show this is ≥ 0

(1) • $C_t^* \in \arg\max U(C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_t - C_t) + \tilde{Y}_{t+1}\right)\right]$

$\Rightarrow V^t(K_t+\theta) - V^t(K_t) \ge \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_t+\theta-C_t)+\tilde{Y}_{t+1}\right) - V^{t+1}\left(\tilde{R}_{t+1}(K_t-C_t)+\tilde{Y}_{t+1}\right.\right.$

$\ge 0 \Leftrightarrow$ need $V^{t+1}(\cdot)$ non-decreasing to prove V^t is non-decreasing.

$V^{T+1}(K_{T+1}) \equiv 0$

Now for concavity

always above secant line

$V^t(\theta K_1 + (1-\theta)K_2) - \theta V^t(K_1) - (1-\theta)V^t(K_2) \ge 0, \ \forall K_1, K_2, \ \forall\theta\in[0,1]$

Preservmax Theorem

$V^t(\theta K_1 + (1-\theta)K_2) - \theta V^t(K_1) - (1-\theta)V^t(K_2) = \max_{X_t\in\mathcal{X}(\theta K_1+(1-\theta)K_2)} U(\theta K_1+(1-\theta)K_2, C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(\theta K_1+(1-\theta)K_2\right)\right.$ -

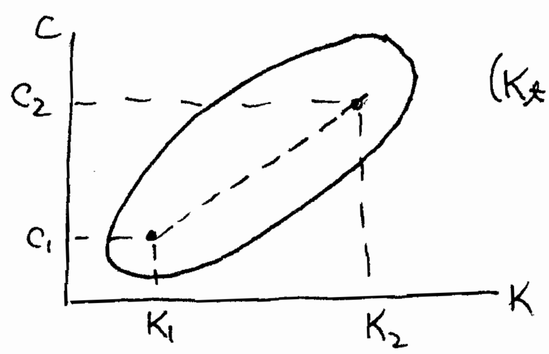
$c_1 \in \arg\max \Rightarrow \quad + \tilde{Y}_t)\Big] - \left(\theta \max_{C_t\in\mathcal{X}(K_1)} U(K_1, C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_1 - C_t) + \tilde{Y}_{t+1}\right)\right]\right)$

$c_2 \in \arg\max \Rightarrow \quad - \left[(1-\theta) \max_{C_t\in\mathcal{X}(K_2)} U(K_2, C_t) + \beta E_t\left[V^{t+1}\left(\tilde{R}_{t+1}(K_2 - C_t) + \tilde{Y}_{t+1}\right)\right]\right]$

Q_1

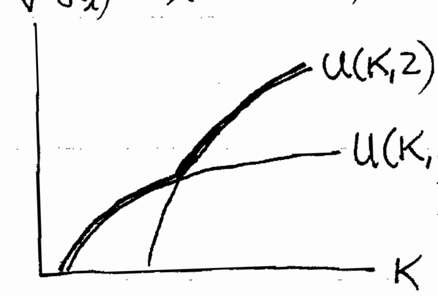
$$\geq u(\theta K_1+(1-\theta)K_2,\ \theta C_1+(1-\theta)C_2) - \theta\{u(K_1,C_1) - (1-\theta)u(K_2,C_2)$$
$$+\beta E_t\Big[V^{t+1}\big(\tilde R_{t+1}(\theta(K_1-C_1)+(1-\theta)(K_2-C_2))+\tilde y_{t+1}\big)\Big\} - \theta V^{t+1}\big(\tilde R_{t+1}(K_1-C_1)+\tilde y_{t+1}\big)$$
$$-(1-\theta)V^{t+1}\big(\tilde R_{t+1}(K_2-C_2)+\tilde y_{t+1}\big)\Big] \geq 0$$

need $\theta C_1 + (1-\theta)C_2 \in [C_1, C_2]$



$(K_t, C_t) \in \mathcal{W}$, $\mathcal{W}$ is a convex set

Discrete choices make you look risk loving



$\overset{t+1}{V}(\cdot)$ concave assumed gives $\beta E_t[\circ] \geq 0$
$u(\cdot)$ jointly concave in (C,K). (think baseball cap)

Bellman's equation for diffusion process in continuous time (p 37)

$$\rho V^t(K_t) - \underset{\frac{\partial V^t}{\partial t}}{V^t_t} = \max_{X_t \in \mathcal{X}(K_t)} U^t(K_t,X_t) + V^t_K(K_t)\,A(K_t,X_t) + V^t_{KK}(K_t)\frac{\Omega(K_t,X_t)}{2}$$

lim of $K_{t+h} = K_t + hA(K_t,X_t) + \sqrt{h\,\Omega(K_t,X_t)}\ \tilde\varepsilon_{t+1}$

$\uparrow$ mean 0, variance 1

in particular, it is often most convenient to take limit w/ $\varepsilon_{t+h} = \begin{cases} +1 & \text{w prob. } 1/2 \\ -1 & \text{w prob. } 1/2 \end{cases}$

$dK = A(K_t,X_t)\,dt + \sqrt{\Omega(K_t,X_t)}\,dz$

$dz \sim$ mean 0, variance dt

$$\max \ E_0\left[\int_0^t e^{-\rho t} u(K_t, X_t)\,dt\right]$$
$$\text{s.t. } dK = A(K_t, X_t)\,dt + \sqrt{\Omega(K_t, X_t)}\,dz$$

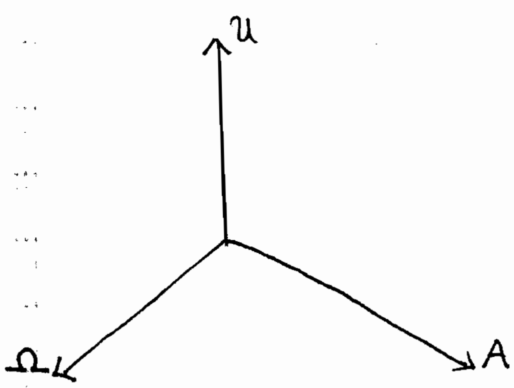
$$V^t(K_t) = \max \ u(K_t,X_t) + e^{-\rho h} E\left[V^{t+h}(K_{t+h})\right] \cong \cancel{V^t(K_t) + \left[V^{t+h}(K_t) - V^t(K_t)\right]}$$
$$\approx V^t(K_t) + (e^{-\rho h} - 1)V^t(K_t) + e^{-\rho h}\left[V^{t+h}(K_t) - V^t(K_t)\right]$$
$$+ e^{-\rho h}\left[V^{t+h}(K_t + hA(K_t, X_t)) - V^{t+h}(K_t)\right]$$
$$+ e^{-\rho h} E_t\left[V^{t+h}(K_t + hA(K_t,X_t) + \sqrt{h\,\Omega(K_t,X_t)}\,\tilde{\varepsilon}_{t+1}) - V^{t+h}(K_t + hA(K_t,X_t))\right]$$

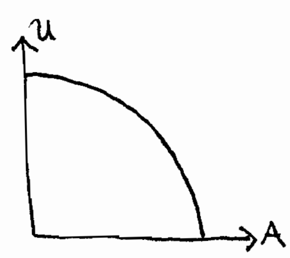
2.15 Hint: Prob. 1 : concavity : ~~might~~ think of X as amount of labor ≈ linear function of capital

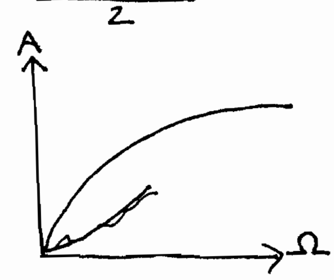
convexity: think of X as labor to capital ratio held constant

redefine X and use change of variables. Careful then about last question.

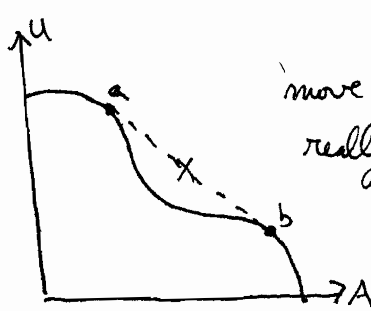
$$\rho V^t(K_t) - V_t^t(K_t) = \max_{X_t} u(K_t, X_t) + V_K^t(K_t)\, A^t(K_t, X_t) + V_{KK}^t \frac{\Omega^t(K_t, X_t)}{2}$$



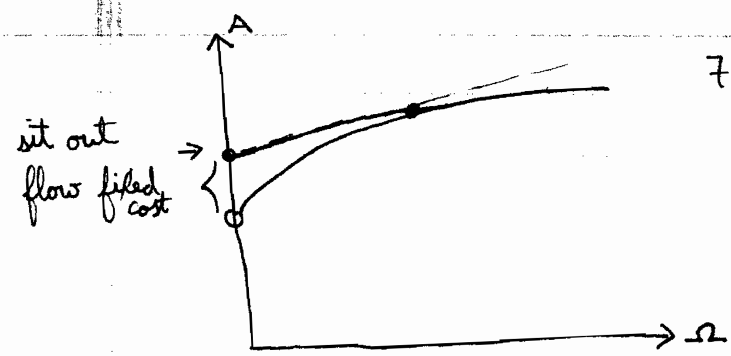




Chattering principle:



move back + forth between a & b really quickly in continuous time ⇒ gets you convexity



∃ flow fixed cost

can get convex hull by chattering

Symmetry Method

Theorem: one version is a Boyd Chapter

Symmetry of constraint set
Symmetry of preferences $\Rightarrow$ Symmetry of the value function

~~Symmetry of~~

$$V^t(K_t) = \max_{X \in \mathcal{X}(K_t)} \psi\left(K_t, X_t, E\left[V^{t+h}\left(\Gamma(K_t, X_t, \tilde{\omega}_{t+h})\right)\right]\right)$$

$V_t = \psi(K_t, X_t, E[V_{t+h}])$ $\quad V_{T+h} \equiv 0$

$\mathcal{X}(\cdot)$ contemp constraint

$\Gamma(\cdot)$ Transition eq.

A: Contemporaneous Constraints: $(K_t, X_t) \in \mathcal{W} \Leftrightarrow T(K_t, X_t) \in \mathcal{W}$

$(\hat{K}_t, \hat{X}_t) = T(K_t, X_t)$ $\quad T(\cdot)$ is a triangular transformation

$\hat{K}_t = T^K(K_t)$ $\quad \hat{X}_t = T^X(K_t, X_t)$

B: Intertemporal Constraints: $\{ K_{t+h} = \Gamma(K_t, X_t, \tilde{\omega}_{t+h}) \Leftrightarrow \underbrace{T(K_{t+h})}_{\hat{K}_{t+h}} = \Gamma(\underbrace{T^K(K_t)}_{\hat{K}_t}, \underbrace{T^X(K_t, X_t)}_{\hat{X}_t}, \tilde{\omega}_{t+h})$

transformation must be invertible

Symmetry of Preferences

Def. S is a preference symmetry corresponding to T if S is monotonically increasing

iff $\quad V_t=\Psi(K_t, X_t, E[V_{t+h}]) \Leftrightarrow \underset{\hat{V}_t}{S(K_t,V_t)}=\Psi\Big(\overset{\hat{K}_t}{T^K(K_t)}, \overset{\hat{X}_t}{T^X(K_t,X_t)}, E\big[\underset{\hat{V}_{t+h}}{S(\Gamma(K_t,X_t,\omega_{t+h}),V_{t+h})}\big]\Big)$

S describes what $T(\cdot)$ does to the utility function

If you have symmetry of constraint sets and symmetry of preferences, then
$V(T(K_t))=S(V(K_t))$

2.20 Examples of symmetry method

Merton Model: $\quad \overset{V_t(W)=}{\max_{C,\alpha}} \; E_0\left[\int e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right]$

s.t. $\quad dW=[rW+\alpha\mu-C_t]dt + \alpha\sigma dz \quad , dz=\sqrt{dt}\,\tilde{\varepsilon}$

$W(0)=W_0 \qquad \Delta Z=\pm\sqrt{h}$

$\mu=$ E return stock $-$ E return T Bill $\cong \frac{0.01}{YR}$

$\sigma^2=0.0225/yr$

$$\rho V(W,t)-\overset{\frac{dV}{dt}}{V_t(W,t)}=\max_{C,\alpha}\frac{C^{1-\gamma}}{1-\gamma}+V_W(W,t)[rW+\alpha\mu-C]+V_{WW}(W,t)\frac{\alpha^2\sigma^2}{2}$$

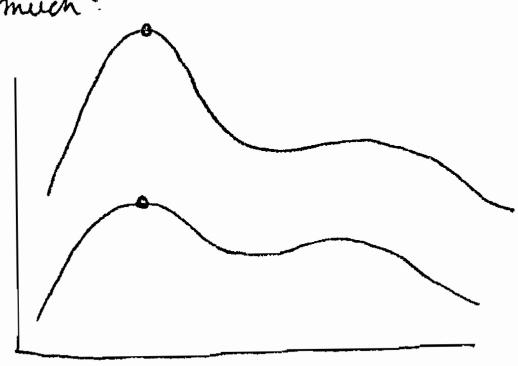
quadratic at a point in time

$C\to\theta C \qquad V\to\theta^{1-\gamma}V$

what would allow us to consume θ as much?

$W\to\theta W, \quad \alpha\to\theta\alpha$

$\Rightarrow V(\theta W,t)=\theta^{1-\gamma}V(W,t)$



In particular, let $\theta = \frac{1}{W}$, $V(1,t) = \left(\frac{1}{W}\right)^{1-\gamma} V(W,t)$

$$\Rightarrow V(W,t) = W^{1-\gamma} V(1,t)$$

$$V_W(W,t) = (1-\gamma) W^{-\gamma} V(1,t)$$
$$V_{WW} = -\gamma(1-\gamma) W^{-\gamma-1} V(1,t)$$

$$\Rightarrow \rho W^{1-\gamma} V(1,t) - W^{1-\gamma} V_t(1,t) = \max_{c,\alpha} \frac{c^{1-\gamma}}{1-\gamma} + (1-\gamma) W^{-\gamma} V(1,t)\left[rW + \alpha\mu - c\right] - \gamma(1-\gamma) W^{-\gamma-1} V(1,t) \frac{\alpha^2\sigma^2}{2}$$

$$c^{-\gamma} = (1-\gamma) W^{-\gamma} V(1,t)$$
$$c = \underbrace{\left[(1-\gamma)V(1,t)\right]^{-\frac{1}{\gamma}}}_{\text{average propensity to consume}} W$$

$$\frac{c}{W} = A(t)$$

$$\frac{A(t)^{-\gamma}}{1-\gamma}$$

Divide by $\frac{W^{1-\gamma}}{1-\gamma}$

$$\mu V_W + V_{WW} \alpha \sigma^2 = 0$$
$$\alpha = \frac{\mu}{-\frac{V_{WW}}{V_W}\sigma^2} = \frac{\mu/\sigma^2}{\text{absolute risk aversion of } V} = \frac{\mu}{\gamma\sigma^2}$$

ARA of $V = \frac{\gamma}{W}$
RRA of $V = \gamma$

$$\Rightarrow \rho A^{-\gamma} + \gamma \dot{A}^{-\gamma-1} = A^{1-\gamma} + (1-\gamma) A^{-\gamma}\left[r + \frac{\alpha}{W}\mu - A\right] - \gamma$$
$$= A^{1-\gamma} + (1-\gamma) A^{-\gamma}\left[r + \frac{\mu^2}{\gamma\sigma^2} - A\right] - (1-\gamma)\gamma A^{-\gamma}\left(\frac{\mu}{\gamma\sigma^2}\right)^2 \frac{\sigma^2}{2}$$

This is a ordinary differential equation in A.
Solve by writing $B = \frac{1}{A}$.

$$\max_{\alpha_t, C_t} E_t\left[\int_t^\infty e^{-\rho t} \log(C_t)\, dt\right]$$
$$\text{s.t. } dW = \left[r_t W_t + \alpha_t \mu_t - C_t\right] dt + \alpha_t \sigma_t dZ$$

$C \to \theta C$

$W \to \theta W$

$\alpha \to \theta \alpha$

$$v \to v + \frac{1 - e^{-\rho T}}{\rho} \log(\theta)$$

$$V(\theta W,t) = V(W,t) + \frac{1-e^{-\rho T}}{\rho}\log(\theta)$$

Let $\theta = \frac{1}{W} \Rightarrow V(1,t) = V(W,t) - \frac{1-e^{-\rho T}}{\rho}\log(W)$.

$$\Rightarrow V(W,t) = V(1,t) + \frac{1-e^{-\rho T}}{\rho}\log(W)$$

$$\frac{1}{C} = \frac{1-e^{-\rho T}}{\rho}W^{-1} \Rightarrow \frac{C}{W} = \frac{\rho}{1-e^{-\rho T}}$$

$$\text{MAX} \quad E_t\left[\int_t^T e^{-\rho t}\frac{e^{-aC}}{-a}dt\right] \qquad \text{s.t. } dW = [rW + \alpha\mu - c]dt + \alpha\sigma dz$$

$$C \to C + \theta \qquad V \to e^{-a\theta}V$$

let r be constant

$W \to W + \frac{\theta}{r}$

$\alpha \to \alpha$

$$V\left(W+\frac{\theta}{r},t\right) = e^{-a\theta}V(W,t)$$

let $\theta = -rW$

$$V(0,t) = e^{arW}V(W,t) \quad \Rightarrow V(W,t) = e^{-arW}V(0,t)$$

Capitalization symmetries $(V \to V)$

noncapital income

$$V(W,Y) = \max \int_t^\infty e^{-rt}u(c_t)dt$$

$$dW = [rW + \alpha\mu + Y - c]dt + \alpha\sigma dz.$$

$W \to W + \frac{\theta}{r} \quad \left(W \to W + \int_t^\infty e^{-rt}Y dt\right)$

$Y \to Y - \theta$

$V(W+\frac{\theta}{r}, Y-\theta, t) = V(W,Y,t)$

Let $\theta = Y \Rightarrow V(W+\frac{Y}{r}, 0, t) = V(W,Y,t)$ capitalized labor income

$\alpha = \frac{\mu}{\gamma\sigma^2}(W+\frac{Y}{r})$

CRTS

competitive output & factor markets

CRTS adjustment cost function

$\Rightarrow V(\theta K) = \theta V(K)$ $\theta = \frac{1}{K}$ $\Rightarrow V(1) = \frac{1}{K}V(K)$

$\Rightarrow V(K) = KV(1)$

$\frac{V(K,t)}{K} = V(1,t) =$ avg q = marginal q = $V_K(K,t) = V(1,t)$

Chris Carrol Habit formation $\Rightarrow$ can place \$ value of habit burden (Joe Lupton)

Horizontal & Vertical Aggregation

Legendre Conjugate functions

$V(W_t, Z_t) = \max_{c, S_i} \; u(c_t) + \sum_i \pi_i(Z_t)\beta E_t\left[V_{t+1}(R_i(Z_t)S_i + Y_i(Z_t, \tilde{\omega}_{t+1}); \Gamma_i(Z_t, \ldots\right.$

s.t. $c + \sum_i S_i = W$

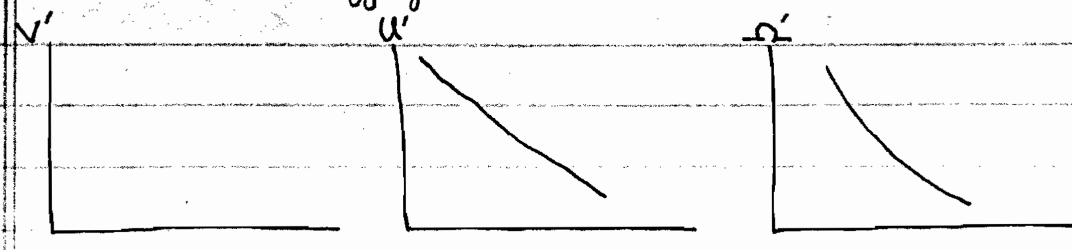
Lagrange: $+\lambda\left[W - c - \sum_i S_i\right]$

$V(W_t, Z_t) - \lambda W \Leftrightarrow \underbrace{\max_c \; u(c_t) - \lambda c}_{-u^*(\lambda)} + \sum_i \underbrace{\left(\max_{S_i} \pi_i(z)\beta E_t[V_t(\cdot)] - \lambda S_i\right)}_{-\Omega^*(\lambda)}$

read $\approx$ p.70 2005 notes

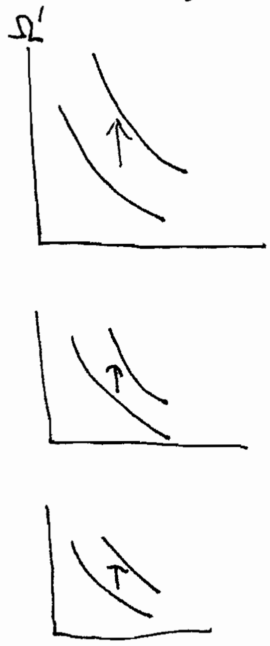
2.22

Horizontal + Vertical aggregation



V' horizontal sum of U', Ω' (if U', Ω' downward sloping, V' is too)

$\beta \uparrow$



shifting up everywhere $\Leftrightarrow$ shifting right everywhere

anything that depends on sign of derivative, preserved under vertical aggregation.

if $V_{t+1}'''(K_{t+1}) \geq 0 \ \forall K_{t+1} \Rightarrow E_t\left[V'''(\tilde{R}S + \tilde{y})\right] \geq 0$

$$-\frac{V''(K_{t+1})}{V'(K_{t+1})} \geq a \quad \Leftrightarrow E_t\left[V_{t+1}''(K_{t+1}) - a V_{t+1}'(K_{t+1})\right] \geq 0$$

$$-\frac{\Omega''}{\Omega'} = \frac{-E[V_{t+1}'']}{E[V_{t+1}']} \geq a$$

$$-K_{t+1} V''(K_{t+1}) - \gamma V'(K_{t+1}) \geq 0 \qquad F(K) = V(RK) \Rightarrow F'(K) = R V'(RK)$$

$$F''(K) = R^2 V''(RK)$$

$$KF''(K) = R^2 K V''(RK)$$

$$K \to K + \tilde{\varepsilon} \Rightarrow E\left[(K+\tilde{\varepsilon}) V''(K+\tilde{\varepsilon}) - \gamma V'(K+\tilde{\varepsilon})\right] \geq 0$$

if $\Omega(K) = E\left[V(K+\tilde{\varepsilon})\right]$, want $-K\Omega''(K) - \gamma \Omega'(K)$

$$-K\, E\left[V''(K+\tilde{\varepsilon})\right] - \gamma E\left[V'(K+\tilde{\varepsilon})\right]$$

$$E\left[(K+\tilde{\varepsilon}) V''(K+\tilde{\varepsilon})\right] \geq E\left[K V''(K+\tilde{\varepsilon})\right] \text{ if } E\tilde{\varepsilon} = 0$$

$E[\tilde{\varepsilon}\, V''(K+\tilde{\varepsilon})] \geq 0$

$\underset{0}{E[\tilde{\varepsilon}]}\, E[V''(K+\tilde{\varepsilon})] + Cov(\tilde{\varepsilon}, V''(K+\tilde{\varepsilon}))$

$\Rightarrow V'''(\cdot) \geq 0$

$V'''V' - (V'')^2 \geq 0$ DARA

$V_{KK}V_{PP} - (V_{KP})^2 \geq 0$

$\begin{bmatrix} V''' & V'' \\ V'' & V' \end{bmatrix}$ *positive definite* $\begin{bmatrix} V_{KK} & V_{KP} \\ V_{KP} & V_{PP} \end{bmatrix}$ *neg def.*

Vertical aggregation often straightforward.

$$V^t(W_t, Z_t) = \max_{\substack{C, S_i \\ C+\sum_i S_i = W_t}} \mathcal{U}^t(C) + \underbrace{\sum_i \pi_i(Z_t)\beta E_t\left[V^{t+1}\left(R_i(Z_t)S_i + Y_i(Z_t, \tilde{\omega}_{t+1}), \beta, \Gamma^i(Z_t, \tilde{\omega}_{t+1})\right)\right]}_{\Omega^{i,t}(S_i, \beta, Z_t)}$$

$$= \max_{\substack{C, S_i \\ C+\sum_i S_i = W_t}} \mathcal{U}^t(C) + \sum_i \Omega^{i,t}(S_i, \beta, Z_t)$$

$$= \max_{C, S_i} \mathcal{U}^t(C) + \sum_i \Omega^{i,t}(S_i, \beta, Z_t) + \lambda\left[W_t - C - \sum_i S_i\right]$$

envelope theorem: $V_W = \lambda$

$$\max_{W_t} V(W_t, \beta, Z_t) - \lambda W_t = \max_C (\mathcal{U}(C) - \lambda C) + \sum_i \max_{S_i}(\Omega^{it}(S_i, \beta, Z_t) - \lambda S_i)$$

assume concavity (Rockafellar's Convex Analysis)

$$= -\mathcal{U}^*(\lambda) - \sum_i \Omega^{i*}(\lambda, \beta, Z_t)$$

$\Omega^*, \mathcal{U}^*$ = Legendre conjugate functions

$$V^*(\lambda, p) = -\max_W V(W, p) - \lambda W = \lambda \underbrace{V_W^{-1}(\lambda, p)}_{W} - V(V_W^{-1}(W, p), p)$$

$\Omega_S = \lambda$, $\mathcal{U}_C = \lambda$

$$= \lambda \underbrace{\mathcal{U}_C^{-1}(\lambda)}_{C} - \mathcal{U}(\mathcal{U}_C^{-1}(\lambda)) + \sum_i \lambda \overbrace{\Omega_{S_i}^{-1}(\lambda, p)}^{\Sigma S_i} - \Omega^i(\Omega_{S_i}^{-1}(\lambda, p), p)$$

$$\left(V^*(\lambda,p)\right)^* = V(w,p)$$

$$V^*(\lambda,p) = \lambda V_w^{-1}(\lambda,p) - V(V_w^{-1}(\lambda,p)\},p)$$

$$V^{**}(w) = \min_\lambda \lambda w - V^*(\lambda) = \min_\lambda \lambda w - \underbrace{\lambda V_w^{-1}(\lambda)}_{\omega} + V(\underbrace{V_w^{-1}(\lambda,p)}_{\omega},p)$$

$$= \min_\omega \overset{V'(\omega)}{\cancel{\lambda}} w - \overset{V'(\omega)}{\cancel{\lambda}}\omega + V(\omega)$$

$$\Rightarrow \cancel{V'(\omega) = \lambda} \quad V''(\omega)w - V''(\omega)\omega - V'(\omega) + V'(\omega) = 0$$

$$\Rightarrow \omega = w \text{ so long as } V'' \neq 0$$

$$w = V_w^{-1}(\lambda,p)$$

$$V_w(w,p) = \lambda \qquad V_{ww}(w,p)dw + V_{wp}(w,p)dp = d\lambda$$

$$dw = \frac{d\lambda}{V_{ww}} - \frac{V_{wp}}{V_{ww}}$$

$$\frac{\partial}{\partial\lambda} V_w^{-1}(\lambda,p) = \frac{1}{V_{ww}\left(V_w^{-1}(\lambda,p)\right)} \qquad \frac{\partial}{\partial p} V_w^{-1}(\lambda,p) = \frac{-V_{wp}(V_w^{-1}(\lambda,p),p)}{V_{ww}(V_w^{-1}(\lambda,p),p)}$$

$$V^*(\lambda,p) = -\max_w\left(V(w,p) - \lambda w\right) = \lambda V_w^{-1}(\lambda,p) - V(V_w^{-1}(\lambda,p),p)$$

$$V_\lambda^*(\lambda,p) = V_w^{-1}(\lambda,p) = w$$

$$\lambda = V_w(w) \qquad w = V_\lambda(\lambda)$$

$$V(w,p) = -\max_\lambda\left(V^*(\lambda,p) - \lambda w\right) = wV_\lambda^{*-1}(w,p) - V^*(V_\lambda^{*-1}(\lambda,p),p)$$

$$V_w(w,p) = V_\lambda^{*-1}(w,p) = \lambda$$

$$V_\lambda^* = w \qquad (w = V_w^{-1}(\lambda,p)) \qquad V_p^* = -V_p$$

$$V_{\lambda\lambda}^* = \frac{1}{V_{ww}} \qquad V_{p\lambda}^* = -V_{pw}/V_{ww}$$

$$V_{\lambda\lambda\lambda}^* = \frac{-V_{www}}{V_{ww}^3}$$

$V_W = \lambda$ $\qquad \lambda = (V_\lambda^{*})^{-1}(W,\rho)$ $\qquad V_\rho = -V_\rho^*$

$V_{WW} = \dfrac{1}{V^*_{\lambda\lambda}}$ $\qquad V_{\rho W} = -V^*_{\rho\lambda}/V^*_{\lambda\lambda}$

$V_{WWW} = -\dfrac{V^*_{\lambda\lambda\lambda}}{V^{*2}_{\lambda\lambda}}$

$V_{\rho W} \geq 0 \Leftrightarrow V^*_{\rho\lambda} \geq 0$
assuming concavity

$$-\frac{V_{KK}\cdot K}{V_K} \geq \gamma \qquad (K=W)$$

$$-\frac{V^*_\lambda}{\lambda V^*_{\lambda\lambda}} \geq \gamma$$ relative risk tolerance of conjugate function

$$V^* = u^* + \sum_i \Omega^{i*}$$

$$\bar{V}(K,Z) = \Pi(K,Z) + \max_{\substack{I \geq 0 \\ S=(1-\delta)K+I}} \Phi(I,Z) + \overbrace{E_t\left[D(\beta, Z_{t+1}) V^{t+1}(S,\beta,Z_{t+1})\right]}^{\Omega(S,\beta,Z)}$$

$$\Leftrightarrow \underbrace{V(K,Z) - \Pi(K,Z)}_{F(K,Z)} = \max_I \Phi(I,Z) + E_t[\cdot] + \lambda\left[(1-\delta)K + I - S\right]$$

ex profit value

$$\max_K \left(F(K,Z) - \lambda(1-\delta)K\right) = \max_I\left(\Phi(I,Z) + \lambda I\right) + \max_S\left(\Omega(S,\beta,Z) - \lambda S\right)$$

$$-F^*((1-\delta)\lambda, Z) = -\Phi^*(-\lambda, Z) - \Omega^*(\lambda,\beta,Z)$$

$$\Leftrightarrow F^*((1-\delta)\lambda, Z) = \Phi^*(-\lambda, Z) + \Omega^*(\lambda,\beta,Z)$$

Extra Credit: max of X : get S times conjugate function

$$F^*(\lambda(1-\delta), Z) = \Omega^*(\lambda + \Phi^*(-\lambda, Z), Z)$$