ECON 610 - STOCHASTIC DYNAMIC OPTIMIZATION

3.6.2006 · ~~NEXT WEEK~~ WEDNESDAY AND SO FORTH, MEETING 12:40 - 2:00

MAX $\int_0^T f(t, x(t), u(t))\,dt$ $\quad$ $x(t)$ state variable

s.t. $\dot{x}(t) = g(t, x(t), u(t))$ $\quad$ $u(t)$ control variable

$x(0) = x_0$

Hamiltonian : $\mathcal{H} \equiv f(t, x(t), u(t)) + \lambda(t) g(t, x(t), u(t))$ $\quad$ $\lambda(t)$ multiplier

Necessary conditions : Let optimal path be $x^*(t), u^*(t)$

(1) $\frac{\partial \mathcal{H}}{\partial u} = 0$ $\quad$ $[f_u(t, x^*, u^*) + \lambda(t) g_u(t, x^*, u^*) = 0$ for all $0 \le t \le T]$

(2) $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}$ $\quad$ $-[f_x(t, x^*, u^*) + \lambda(t) g_x(t, x^*, u^*)] = \dot{\lambda}(t)$

$\lambda(T) = 0$

(3) $\dot{x}(t) = g(t, x(t), u(t))$

$x(0) = x_0$

Special case : MAX $\int_0^T f(t, x(t), \dot{x}(t))\,dt$ $\quad$ Calculus of Variations

$u = \dot{x}$ $\quad$ s.t. $x(0) = x_0$

Euler equation : $f_x(t, x(t), \dot{x}(t)) = \frac{d}{dt}\left(f_{\dot{x}}(t, x(t), \dot{x}(t))\right)$

$\mathcal{H} = f(t, x, \dot{x}) + \lambda \dot{x}$

$\frac{\partial \mathcal{H}}{\partial u} = f_{\dot{x}} + \lambda = 0$

$\lambda = -f_{\dot{x}}$

$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} = -f_x$

Demand: $P(t) = a - bQ(t)$, $Q(t)$ = quantity sold

MAX $\int_0^T e^{-rt} p(t) q(t)\, dt$

s.t. $\int_0^T q(t)\, dt = B$ fixed resources

$q(t) \geq 0$

Define $z(t)$ the amount sold up to time t, i.e., $z(t) = \int_0^t q(\tau)\, d\tau$.

$\Rightarrow q(t) = \dot{z}(t)$

$\Rightarrow$ MAX $\int_0^T e^{-rt} [a - b\dot{z}(t)] \dot{z}(t)\, dt$

s.t. $z(0) = 0$, $z(T) = B$

Calculus of variations $\Rightarrow$ Euler equation $f(t, z, \dot{z}) = e^{-rt}[a\dot{z} - b(\dot{z})^2]$

$f_z = 0$

$f_{\dot{z}} = (a - 2b\dot{z})e^{-rt}$

$\Rightarrow 0 = \frac{d}{dt}\left\{[a - 2b\dot{z}]e^{-rt}\right\}$

$\Leftrightarrow e^{-rt}[a - 2b\dot{z}(t)] = k$ constant

$\Rightarrow \dot{z}(t) = \frac{e^{rt}k - a}{-2b} = \frac{a - e^{rt}k}{2b}$

Integrate and use boundary conditions

Sufficient conditions: • $f(t, x, u)$ and $g(t, x, u)$ concave in (x, u)

• $\lambda(t) \geq 0$

If $g(t, x, u)$ linear in (x, u), we don't need $\lambda(t) \geq 0$.

$\mathcal{X}$, $\Gamma(x) \subset \mathcal{X}$ compact and convex

$F(x,y)$

$$V(x) = \max \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t) \quad , t = 0,1,2,\ldots$$

$$0 < \beta < 1$$

$$x_0 = x \text{ given}$$

$V(\cdot)$ value function

- Assume $F(\cdot)$ concave. Then $V(\cdot)$ concave.

Bellman equation: $V(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}$

$V(x)$ $V(y)$

x y

0 1 2 3 4 ...

Notice $F(\cdot)$ time independent, $\mathcal{X}$ and $\Gamma(x)$ are as well. Then optimal solution is time independent. Assuming strict concavity, unique maximum achieved at $y = g(x)$. Call $g(\cdot)$ the optimal policy function.

Claim: V' exists and $V'(x) = F_x(x, g(x))$ for $x > 0$.

Proof: $x > 0$ and $\Delta x > 0$

$$V(x+\Delta x) \geq V(x) + F(x+\Delta x, g(x)) - F(x, g(x))$$

$$\Leftrightarrow \quad \frac{V(x+\Delta x) - V(x)}{\Delta x} \geq \frac{F(x+\Delta x, g(x)) - F(x, g(x))}{\Delta x}$$

$$\lim_{\Delta x \to 0^+} \quad V'(x) \geq F_x(x, g(x))$$

Now let $x > 0$, $\Delta x < 0 \Rightarrow V(x) - V(x-\Delta x) \geq F(x, g(x)) - F(x-\Delta x, g(x))$

$$\Leftrightarrow \quad \frac{V(x) - V(x-\Delta x)}{\Delta x} \leq \frac{F(x, g(x)) - F(x-\Delta x, g(x))}{\Delta x}$$

$$\lim_{\Delta x \to 0^-} \quad V'(x) \leq F_x(x, g(x))$$

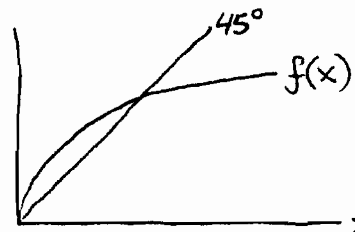
For any concave function $h(x)$, $h'_+(x) \leq h'_-(x)$

Then $V'(x) = F_x(x, g(x))$.

Back to Bellman equation: $F_y(x, g(x)) + \beta V'(g(x)) = 0$

$$\Rightarrow F_y(x, g(x)) + \beta F_x(g(x), g(g(x))) = 0$$

Example: $c_t + x_{t+1} = f(x_t)$



$c_t = f(x_t) - x_{t+1}$

$u(c_t)$ utility function, $u' > 0$, $u'' < 0$, $u'(0) = \infty$

$$V(x) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $x_0 = x$

$c_t = f(x_t) - x_{t+1}$, $t = 0, 1, 2, \dots$

$$V(x) = \max \sum_{t=0}^{\infty} \beta^t u(f(x_t) - x_{t+1})$$

s.t. $x_0 = x$

Bellman equation $\quad V(x) = \max_{0 \le c_0 \le f(x)} \{u(c_0) + \beta V(f(x) - c_0)\}$

$c_0^* = g(x)$

First order condition $\quad u'(c_0^*) - \beta V'(f(x) - c_0^*) = 0$

$V'(x) =$

Try $u(c) = \log(c)$, $f(x) = Ax^\alpha$, $\alpha \in (0,1)$, $A > 0$

3.8 $$V(z_0)=\max_{\{c_t\}}\sum_{t=0}^{\infty}\beta^t u(c_t) \quad , u'>0 \text{ and } u''<0$$

$$\text{s.t. } c_t+x_t=z_t$$

$$z_t=f(x_{t-1}) \quad , f'>0 \text{ and } f''<0$$

$$z_0>0 \text{ given}$$

Bellman equation: $V(z_0)=\max_{0\leq c_0\leq z_0}\{u(c_0)+\beta V(z_1)\}$

$z_1=f(z_0-c_0)$

Optimality conditions: $u'(c_0^*)-\beta f'(z_0-c_0^*)V'(z_1)=0$

$c_0^*=g(z_0)$ optimal policy function

$V'(z)=u'(g(z))$ for all $z>0$

$$\Rightarrow u'(g(z))=\beta f'(z-g(z))u'(g(f(z-g(z)))) \text{ functional equation in } g(z)$$

Example: $u(c)=\log(c)$; $f(x)=Ax^\alpha, \alpha\in(0,1), A>0$

Try $g(z)=\lambda z$ because we need $0\leq c\leq z \Leftrightarrow 0\leq g(z)\leq z$.

Then $\frac{1}{\lambda z}=\beta\alpha A(z-\lambda z)^{\alpha-1}\left[\lambda A(z-\lambda z)^\alpha\right]^{-1}$

$\Leftrightarrow \frac{1}{\lambda z}=\alpha\beta A((1-\lambda)z)^{\alpha-1}(\lambda A)^{-1}((1-\lambda)z)^{-\alpha}$

$\Leftrightarrow \frac{1}{z}=\alpha\beta[(1-\lambda)z]^{-1}$

$\Leftrightarrow 1=\alpha\beta(1-\lambda)^{-1}$

$\Leftrightarrow 1-\lambda=\alpha\beta$

$\Leftrightarrow \lambda=1-\alpha\beta$

$\Rightarrow g(z)=(1-\alpha\beta)z$

$c_0^*=(1-\alpha\beta)z_0$ $\qquad c_1^*=(1-\alpha\beta)z_1=(1-\alpha\beta)A(\alpha\beta z_0)^\alpha$ $\qquad$ So on...

$x_0^*=\alpha\beta z_0$ $\qquad x_1^*=\alpha\beta z_1=\alpha\beta A(\alpha\beta z_0)^\alpha$

$z_1^*=A(\alpha\beta z_0)^\alpha$

Euler equations : $u'(c_t^*) = \beta f'(x_t^*) u'(c_{t+1}^*)$ for $t = 0, 1, 2, \ldots$

Stochastic Models, Finite number of states of nature

$i = 1, \ldots, n$ states of nature $\qquad t = 0, 1, 2, \ldots, T$ periods of time <u>left</u>

Reward function $R(i, a)$

$a \in A$, A set of actions

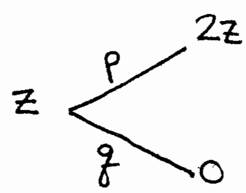
$P_{ij}(a) = \text{Prob}[\text{go from state } i \text{ to state } j \mid \text{action } a \in A]$

$$V_T(i) = \max_{a \in A} \left[R(i,a) + \sum_j P_{ij}(a) V_{T-1}(j) \right]$$

$$\vdots$$

$$V_0(i) = \max_{a \in A} R(i,a)$$

Example: gambling



$p > q$

Let initial wealth be X, and let $T = n$. We want to maximize the log of wealth after n periods.

$V_0(X) = \log(X)$ no more periods / no gambling

$V_1(X) = \max_{0 \leq \alpha \leq 1} p \log(x + \alpha x) + q \log(x - \alpha x)$

$\Rightarrow \alpha^* = p - q \quad \Rightarrow V_1(X) = \log(X) + c$, $c = \log(2) + p \log(p) + q \log(q)$

$V_2(X) = \max_{\alpha} p \log((1+\alpha)x) + q \log((1-\alpha)x) + c$

$\Rightarrow V_2(X) = \log(X) + 2c$

$\vdots$

$V_n(X) = \log(X) + nc$

Discounted Dynamic Programming

$i = 1, 2, \dots$ states of nature (unbounded)

$\{X_n\}_{n=0}^{\infty}$ random variable sequence

Reward function $R(x_n, a_n)$ in period n. Assume $R(\cdot,\cdot)$ continuous and bounded

$$\max_{\{a_n\}_{n=0}^{\infty}} E\left[\sum_{n=0}^{\infty} \alpha^n R(x_n, a_n)\right]$$
$$\text{s.t. } a_n \in A \quad , n = 0, 1, 2, \dots$$

$P\left[X_{n+1} = j \mid X_n = i, a_n \in A\right] = P_{ij}(a_n) \Leftrightarrow$ the rest of the history doesn't matter

We can define a value function that is time independent.

A stationary policy function $f(i) \in A$ for all i is one where we only need to know what state we are currently in, not the entire history, i.e., time independent.

$$V_f(i) = E_f\left[\sum_{n=0}^{\infty} \alpha^n R(X_n, f(x_n)) \mid X_0 = i\right]$$

$V(i) = \sup_f V_f(i)$ for all $i \Rightarrow V(i) = \max_{a \in A}\left[R(i,a) + \alpha \sum_j P_{ij}(a) V(j)\right]$

The maximum is achieved at $f^*(i) = a^*$. Does $V_{f^*}(i) = V(i)$ for all i?

If yes, then $f^*(i)$ is the optimal stationary policy function.

Take a stationary policy function $g(i)$. Let B be the set of functions of $N = \{1, 2, \dots\}$.

Now let $T_g : B \to B$. Then $(T_g v)(i) = R(i, g(i)) + \alpha \sum_j P_{ij}(g(i)) V(j)$.

We can show T_g is monotone: $u \leq v \Rightarrow T_g u \leq T_g v$. Using this operator, applied n times, to v, v converges to V_g as $n \to \infty$. That is, T_g is a contraction mapping. $\lim_{n \to \infty} T_g^n v \to V_g$. $\Rightarrow V(i) = V_{f^*}(i)$ for all i.

3.13 Example: saving under uncertainty

Let k_t be wealth at the beginning of t. Can invest $(k_t - c_t)$ and consume c_t. Flow utility $u(c_t)$, $u'>0$, $u''<0$. Let $\{\tilde{r}_t\}_{t=0}^{\infty}$ be a sequence of random returns, and let k_0 be initial wealth.

$$V(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$\text{s.t. } k_{t+1} = \tilde{r}_t(k_t - c_t), \quad k_0 > 0 \text{ given}$$

Bellman equation: $V(k_0) = \max_{0 \le c_0 \le k_0} \left[u(c_0) + \beta E\left[V(\tilde{r}_0(k_0 - c_0))\right]\right]$

Can show $V'>0$ and $V''<0$. There is a unique maximum $c_0^* = g(k_0)$.

$g(\cdot)$ is time independent because $u(\cdot)$, β, and $V(\cdot)$ are time independent.

$V'(k)$ exists for $k>0$. Then $V'(k) = u'(g(k))$ for $k>0$.

$$u'(c^*) = \beta E\left[\tilde{r} V'(\tilde{r}(k - c^*))\right]$$

(1) $u'(g(k)) = \beta E\left[\tilde{r} u'(g(\tilde{r}(k - g(k))))\right]$ functional equation for $g(k)$

$\Rightarrow u'(c_t^*) = \beta E_t\left[\tilde{r}_t u'(c_{t+1}^*)\right]$, $t = 0, 1, 2, \ldots$ Euler equations

Are these sufficient conditions? No! We need transversality conditions.

(2) $\lim_{t \to \infty} \beta^t E\left[k_t^* u'(c_t^*)\right] = 0$

Why do we need to bother with transversality condition?

Let $(c_t^*, k_t^*)_{t=0}^{\infty}$ be optimal for k_0. Then, for any feasible $(c_t, k_t)_{t=0}^{\infty}$, we have $\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t E\left[u(c_t) - u(c_t^*)\right] \le 0$. We want to show the Euler equation is not sufficient for this to hold.

For any concave function $f(\cdot)$, $f(x)-f(x^*) \leq f'(x^*)(x-x^*)$. Then,

$$E\left[\sum_{t=0}^{T}\beta^t\left(u(c_t)-u(c_t^*)\right)\right] \leq E\left[\sum_{t=0}^{T}\beta^t u'(c_t^*)(c_t-c_t^*)\right]$$

Since $k_{t+1}=\tilde{r}_t(k_t-c_t)$, $c_t = k_t - \frac{1}{\tilde{r}_t}k_{t+1}$, which implies

$$E\left[\sum_{t=0}^{T}\beta^t u'(c_t^*)(c_t-c_t^*)\right] = E\left[\sum_{t=0}^{T}\beta^t u'(c_t^*)\left(k_t-k_t^*-\frac{1}{\tilde{r}_t}\left(k_{t+1}-k_{t+1}^*\right)\right)\right]$$

$$= E\left[\sum_{t=0}^{T}\beta^t \frac{u'(c_t^*)}{\tilde{r}_t}\left(\tilde{r}_t(k_t-k_t^*)-(k_{t+1}-k_{t+1}^*)\right)\right]$$

$\vdots$

$$= E\left[\sum_{t=1}^{T}\beta^{t-1}\frac{k_t-k_t^*}{\tilde{r}_t}\left(\beta\tilde{r}_t u'(c_{t+1}^*)-u'(c_t^*)\right)\right] - \beta^T E\left[\frac{u'(c_T^*)}{\tilde{r}_T}(k_T-k_T^*)\right]$$

$$= -\beta^T E\left[\frac{u'(c_T^*)}{\tilde{r}_T}(k_T-k_T^*)\right] \quad \text{by Euler equation}$$

The transversality condition is need to show $-\beta^T E\left[\frac{u'(c_T^*)}{\tilde{r}_T}(k_T-k_T^*)\right] \leq 0$.

Example: $u(c)=\frac{c^{1-\alpha}}{1-\alpha}$, $\alpha>0 \Rightarrow u'(c)=c^{-\alpha}$

Assume $g(k)=\lambda k$, $\lambda\in(0,1)$

$$u'(g(k))=\beta E\left[\tilde{r}\,u'\left(g\left(\tilde{r}(k-g(k))\right)\right)\right]$$

$$(\lambda k)^{-\alpha}=\beta E\left[\tilde{r}\left(\lambda\tilde{r}(1-\lambda)k\right)^{-\alpha}\right]$$

$$\lambda^{-\alpha}=\beta E\left[\tilde{r}^{1-\alpha}\left(\lambda(1-\lambda)\right)^{-\alpha}\right]$$

$$(1-\lambda)^{\alpha}=\beta E\left[\tilde{r}^{1-\alpha}\right] \qquad \text{~~since $\lambda\in(0,1)$, $(1-\lambda)^\alpha\in(0,1) \Rightarrow$ need~~ for $\alpha>1$}$$

$$\lambda = 1-\left(\beta E\left[\tilde{r}^{1-\alpha}\right]\right)^{\frac{1}{\alpha}} \qquad \text{need } \left(\beta E\left[\tilde{r}^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}\in(0,1) \text{ for feasibility}$$

Does this λ satisfy the transversality condition?

$$k_{t+1}=(k_t-g(k_t))\tilde{r}_t = \tilde{r}_t(1-\lambda)k_t = k_0(1-\lambda)^t\prod_{\tau=0}^{t}\tilde{r}_\tau$$

$$\text{TV: } E\left[\beta^t k_{t+1}^* u'(c_{t+1}^*)\right] = E\left[\beta^t(1-\lambda)^t k_0\left(\prod_{\tau=0}^{t}\tilde{r}_\tau\right)\left(\lambda(1-\lambda)^{t-1}k_0\left(\prod_{\tau=0}^{t-1}\tilde{r}_\tau\right)\right)^{-\alpha}\right]$$

$$=\left(\frac{1-\lambda}{\lambda}\right)^{\alpha}k_0^{1-\alpha}\,E\left[\beta^t(1-\lambda)^{(1-\alpha)t}\,\tilde{r}_t\left(\prod_{\tau=0}^{t-1}\tilde{r}_\tau\right)^{1-\alpha}\right]$$

What happens if $\tilde{r}$ becomes riskier?

$\tilde{X}$ is a mean-preserving-spread of $\tilde{Y}$ if it can be written $\tilde{X}=\tilde{Y}+\tilde{\varepsilon}$, $E[\tilde{\varepsilon}|y]=0$.
Risk averse agents prefer $\tilde{Y}$ to $\tilde{X}$. This is second-order stochastic dominance.

What happens to savings? $s_t=(1-\lambda)k_t$ is optimal savings function

- $\alpha<1 \Rightarrow r^{1-\alpha}$ concave $\Rightarrow E[\tilde{r}^{1-\alpha}]^{\frac{1}{\alpha}} \downarrow$ as $\tilde{r}$ becomes riskier by Jensen's inequality $\Rightarrow (1-\lambda)\downarrow \Leftrightarrow$ less savings
- $\alpha>1 \Rightarrow$ more savings

Suppose $\log(\tilde{r}) \sim N(\mu,\sigma^2)$. Then $E[\tilde{r}]=\bar{r}=\exp(\mu+\frac{1}{2}\sigma^2)$

$$V[\tilde{r}]=\exp(2\mu+\sigma^2)\cdot[e^{\sigma^2}-1]$$
$$=(\bar{r})^2\cdot(e^{\sigma^2}-1)$$

Then $\lambda=1-\left(\beta(\bar{r})^{1-\alpha}\right)^{\frac{1}{\alpha}}\cdot\exp\left(-(1-\alpha)\frac{\sigma^2}{2}\right)$.

$$\frac{d\lambda}{d\sigma^2}=\frac{1-\alpha}{2}\left(\beta(\bar{r})^{1-\alpha}\right)^{\frac{1}{\alpha}}\exp\left(-(1-\alpha)\frac{\sigma^2}{2}\right) \begin{cases} >0 & \text{for } \alpha\in(0,1) \\ <0 & \text{for } \alpha>1 \end{cases}$$

3.15 Reward function $F(x,y)$
Constraint set $\Gamma(x) \subset X$
Previously, we had $\max_{y \in \Gamma(x)} F(x,y)$
Now, let random shocks affect $\Gamma(x)$. Let $\{z_t\} \subset Z$ be a sequence of random variables. So, $\Gamma(x,z) \subset X$. Let $Q(z,dz')$ be probability of moving from z to $z+dz'$.
Take, e.g., $x \subset \mathbb{R}^+$. Then $V(x_0,z_0) = \sup E\left[\sum_{t=0}^{\infty} \beta^t F(x_t,y_t,z_t)\right]$

$$\text{s.t. } y_t \in \Gamma(x_t,z_t)$$

Bellman equation: $V(x,z) = \max_{y \in \Gamma(x,z)} F(x,y,z) + \beta \int V(y,z') Q(z,dz')$

Feasible stationary plan $\pi_t = \pi_t(x,z) \in \Gamma(x,z)$ such that $\pi_t(x_t,z_t) = h(\pi_{t-1}(x_{t-1},z_{t-1}),z_t)$
policy function

Example: production function $f(x,\tilde{z})$, $\tilde{z}$ random variable. (x_0,z_0) given.
$V(x_0,z_0) = \sup E\left[\sum_{t=0}^{\infty} \beta^t u(f(x_t,\tilde{z}_t) - x_{t+1})\right]$

$$\text{s.t. } 0 \leq x_{t+1} \leq f(x_t,\tilde{z}_t),\ t=0,1,2,\ldots$$

Let $f(\cdot)$ be concave and satisfy Inada conditions, and let $u(\cdot)$ be concave.
$\Rightarrow V(\cdot,\cdot)$ concave.

Bellman equation: $V(x_t,z_t) = \max_{0 \leq x_{t+1} \leq f(x_t,\tilde{z}_t)} u(f(x_t,\tilde{z}_t) - x_{t+1}) + \beta E[V(x_{t+1},\tilde{z}_{t+1})]$
Assume $\{\tilde{z}_t\}$ iid. Define $\omega_t = f(x_t,z_t)$. The optimal policy function is $x^*_{t+1} = h(\omega_t)$, and consumption is $g(\omega_t) = \omega_t - h(\omega_t)$.

$$\Rightarrow V(\omega_t) = \max_{0 \leq x_{t+1} \leq \omega_t} u(g(\omega_t)) + \beta E[V(\omega_{t+1})]$$

Then $V'(\omega) = u'(g(\omega))$.
$$u'(g(\omega)) = \beta E\left[f_x(h(\omega), \tilde{z})\, u'(g(f(h(\omega), \tilde{z})))\right]$$

$h(\omega)$ and $g(\omega)$ are increasing.

If you have uncertainty, what happens to the capital stock?
$$x^*_{t+1} = h(f(x^*_t, z_t)) = H(x^*_t, z_t) \quad , \; z_t \sim [\gamma, \delta] \text{ iid} \quad \text{assuming } f_z > 0$$

$x^*_t(z_0, z_1, \ldots, z_{t-1})$ and $c^*_t(z_0, z_1, \ldots, z_{t-1})$ random variables

$F_t(\xi)$ be the CDF of x^*_t. $F_t(\xi)$ converges in distribution to $F^*(\xi)$ as $t \to \infty$.
This $F^*(\xi)$ is the stochastic modified golden rule.

Example: $f(x, z) = \tilde{z} x^\alpha \quad , \; 0 < \alpha < 1 \quad , \tilde{z}_t \sim \text{iid } [\gamma, \delta] \quad , \; 0 < \gamma < \delta < \infty$
$u(c) = \log(c)$
$\Rightarrow$ linear decision rule

TV: $\lim_{t \to \infty} \beta^t E\left[x^*_t u'(c^*_t)\right] = 0$ because capital stock is bounded, consumption is bounded away from 0.

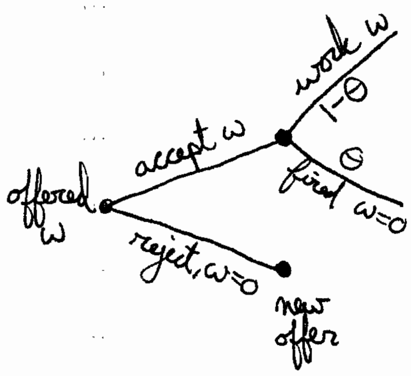
$$g(\omega) = \theta\omega$$

$$\Rightarrow \frac{1}{\theta\omega} = \beta E\left[\alpha\tilde{z}((1-\theta)\omega)^{\alpha-1}\left(\theta\tilde{z}((1-\theta)\omega)^\alpha\right)^{-1}\right]$$
$$\Rightarrow \frac{1}{\theta\omega} = \beta E\left[\frac{\alpha}{\theta(1-\theta)\omega}\right] \quad \Rightarrow \theta = 1 - \alpha\beta$$

$$\Rightarrow g(\omega) = (1-\alpha\beta)\omega$$
$$h(\omega) = \alpha\beta\omega$$

Job Search and Employment

Each period, offered a wage $w \in [0, \bar{w}]$ random variable or not. Can accept. Then, with probability θ receive w in next period and with probability $(1-\theta)$ get fired and receive 0 in next period.



$u(w)$ concave and $u(0)=0$

$w \sim f(w)$ on $[0, \bar{w}]$

$$V(w) = \max\left\{ u(w) + \beta\big((1-\theta)V(w) + \theta V(0)\big),\ u(0) + \beta \int V(w') f(w')\, dw' \right\}$$

We can show $V(w)$ exists by a fixed point theorem.

$\|f\| = \sup_{x \in X} |f(x)|$, X compact

$T: C(X) \to C(X)$, $C(X)$ bounded

$$(Tv)(w) = \max\left\{ u(w) + \beta\big((1-\theta)V(w) + \theta V(0)\big),\ u(0) + \beta \int V(w') f(w')\, dw' \right\}$$

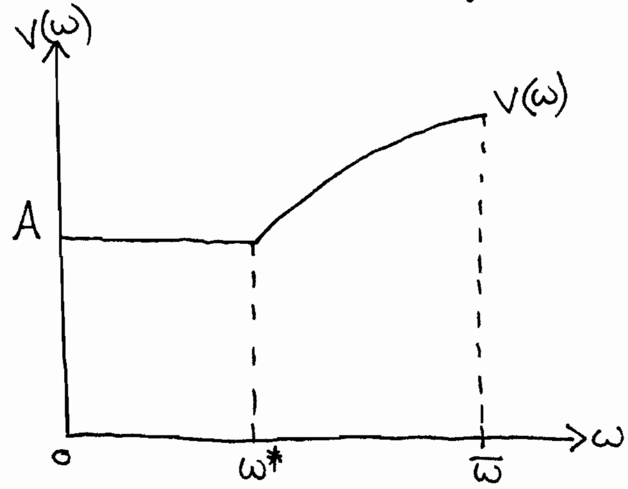
T is a contraction mapping if $\|Tf - Tg\| \le \gamma \|f-g\|$ for all f, g and $0 < \gamma < 1$. $\Rightarrow$ <u>unique</u> fixed point

3.20 Blackwell Theorem: If T satisfies (1) monotonicity, i.e., $f(w) \le g(w) \Rightarrow (Tf)(w) \le (Tg)(w)$ $\forall w$ and (2) discounting, i.e., $\exists\, 0 < \gamma < 1$ such that $T(f+a) \le Tf + \gamma a$ for all f and all $a > 0$.

Here, β plays the role of γ. Monotonicity also holds. The contraction mapping theorem says a unique $V(w)$ exists.

Claim: For a unique $w^* \in [0, \bar{w}]$, $V(w) = \begin{cases} A & \text{if } w \le w^* \\ \frac{u(w) + \beta\theta A}{1 - \beta(1-\theta)} & \text{if } w \ge w^* \end{cases}$

The reservation wage is w^*.



$$A = \beta \int_0^{\bar{w}} V(w') f(w') dw' = \frac{u(w^*)}{1-\beta} = V(0)$$

This is true because the agent must be indifferent to accepting an offer of w^* and rejecting it.

Note: $$A = \beta \int_0^{\bar{w}} V(w') f(w') dw' = \beta \int_0^{w^*} A f(w') dw' + \beta \int_{w^*}^{\bar{w}} \frac{u(w) + \beta\theta A}{1 - \beta(1-\theta)} f(w') dw'$$
$$= \beta A F(w^*) + \frac{\beta^2 \theta A (1 - F(w^*))}{1 - \beta(1-\theta)} + \frac{\beta}{1 - \beta(1-\theta)} \int_{w^*}^{\bar{w}} u(w') f(w') dw'$$

We can show w^* is increasing in β and decreasing in θ.
To do this, manipulating the above yields
$$(1-\beta) A = \frac{\beta}{1 + \beta\theta - \beta F(w^*)} \int_{w^*}^{\bar{w}} u(w') f(w') dw'$$

After some algebra, $(1 - \beta(1-\theta)) u(w^*) = \int_{w^*}^{\bar{w}} (u(w') - u(w^*)) f(w') dw'$.

When is the optimal policy function $g(\cdot)$ concave?
Carroll-Kimball (Econometrica, 1996): when $u(\cdot)$ HARA
$$u(x) = \frac{\alpha}{1-\alpha}\left[\left(A + \frac{x}{\alpha}\right)^{1-\alpha} - 1\right]$$
$$u'(x) = \left(A + \frac{x}{\alpha}\right)^{-\alpha} > 0$$
$$u''(x) = -\left(A + \frac{x}{\alpha}\right)^{-\alpha-1} < 0$$
$$u'''(x) = \left(\frac{1+\alpha}{\alpha}\right)\left(A + \frac{x}{\alpha}\right)^{-\alpha-2} > 0$$

STOCHASTIC CALCULUS

- Consider a random variable X that moves from x_0 to x_t and absorbs iid normal shocks.

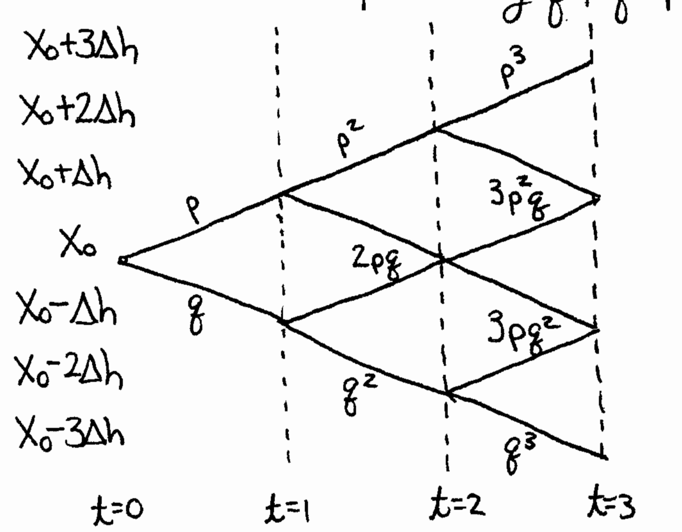
$$E[X_t] = x_0 + \mu t \qquad V[X_t] = \sigma^2 t$$

Let ω be a continuous sequence of random variables such that $d\omega \sim N(0,1)$. This is called a Wiener process.

Then $dx = \mu\, dt + \sigma\, dw$

Take an interval $[0,t]$ and divide it into intervals of length Δt.

$$\Delta X = \begin{cases} +\Delta h & \text{with probability } p \\ -\Delta h & \text{with probability } q, \quad q < p \end{cases} \qquad \Delta X \sim \text{Binomial}(p)$$



Brownian motion is the continuous analogue of this

$$E[X_t] = X_0 + \mu t$$
$$V[X_t] = \sigma^2 t$$

$(p-q)\Delta h \frac{t}{\Delta t}$

Let $n = \frac{t}{\Delta t}$. After n steps, the mean is $n(p-q)\frac{\Delta h}{\cancel{\Delta t}}$, and the variance is $4pq(\Delta h)^2 \cdot n = 4pq(\Delta h)^2 \frac{t}{\Delta t}$

Let $X_0 = 0$, $\Delta h = \sigma\sqrt{\Delta t}$, $p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma^2}\Delta h\right)$, and $q = \frac{1}{2}\left(1 - \frac{\mu}{\sigma^2}\Delta h\right)$.

After n steps, $E[X_t] = (p-q)\Delta h \frac{t}{\Delta t} = \frac{\mu}{\sigma^2}(\Delta h)^2 \frac{t}{\Delta t} = \frac{\mu}{\sigma^2}(\sigma^2 \Delta t)\frac{t}{\Delta t} = \mu t$

and $V[X_t] = 4pq(\Delta h)^2 \frac{t}{\Delta t} = \left(1 - \left(\frac{\mu}{\sigma^2}\Delta h\right)^2\right)(\Delta h)^2 \frac{t}{\Delta t} = \left(1 - \frac{\mu^2}{\sigma^2}\sigma^2 \Delta t\right)\sigma^2 \Delta t \cdot \frac{t}{\Delta t} = (1-\mu^2\Delta t)$

$V[X_t] = (1-\mu^2\Delta t)\sigma^2 t$. As $\Delta t \to 0$, $V[X_t] \to \sigma^2 t$.

What is the length of the path of X?

$E[|\Delta X|] = \Delta h$

$nE[|\Delta X|] = n\Delta h = \frac{t}{\Delta t}\sigma\sqrt{\Delta t}$. As $\Delta t \to 0$, $nE[|\Delta X|] \to \infty$.

This means $\frac{dX_t}{dt}$ doesn't exist.

$dX_t = \mu t + \sigma dw$, $E[dX] = \mu dt$, $V[dX] = \sigma^2 dt$, $dw \sim N(0,1)$

We want to consider a function $y_t = f(x_t)$ and to determine dy.
We can use a Taylor's series approximation around x_0.

$$y_t = f(x_t) \Rightarrow y_0 = f(x_0)$$

$$\begin{aligned}
E[y_t - y_0] &= E[f(x_t) - f(x_0)] \\
&\approx E\left[f'(x_0)(x_t - x_0) + \tfrac{1}{2}f''(x_0)(x_t - x_0)^2 + \cdots\right] \\
&= f'(x_0)(x_0 + \mu t - x_0) + \tfrac{1}{2}f''(x_0)E[(x_t - x_0)^2] + \cdots \\
&= f'(x_0)\mu t + \tfrac{1}{2}f''(x_0)(\sigma^2 t + \mu^2 t^2) + \cdots \\
&= \left[f'(x_0)\mu + \tfrac{1}{2}f''(x_0)\sigma^2\right]t + \tfrac{1}{2}f''(x_0)\mu^2 t^2 + \cdots
\end{aligned}$$

So, $E[dy] = \left(f'(x_0)\mu + \frac{1}{2}f''(x_0)\sigma^2\right)dt$

Similarly, $V[y_t - y_0] = (f'(x_0))^2\sigma^2 t + \cdots$

$V[dy] = (f'(x_0))^2\sigma^2 dt$

* Ito's Lemma: $dy = \left(f'(x_0)\mu + \frac{1}{2}f''(x_0)\sigma^2\right)dt + f'(x_0)\sigma dw$

In $dx = \mu t + \sigma dw$, μ is referred to as the trend and σ as the volatility. In general, $\mu = \mu(x,t)$ and $\sigma = \sigma(x,t)$. What happens to Ito's lemma?

Now, when $\mu = \mu(x)$ and $\sigma = \sigma(x)$, we have a diffusion process.

Suppose $y = F(t, x)$. Then, $dy = \left[F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right]dt + \sigma F_x d\omega$.

Suppose $X = \omega$ (Weiner process: $\mu = 0, \sigma^2 = 1$) $\Rightarrow dy = \left(F_t + \frac{1}{2}F_{\omega\omega}\right)dt + F_\omega d\omega$

Examples:

(1) $dy = y\,d\omega$

$$\text{Regular calculus} \Rightarrow \quad y = e^{\omega}$$
$$\text{Stochastic calculus} \Rightarrow \quad y = \exp\left(\omega - \tfrac{1}{2}t\right)$$

To see this, $F_t = -\frac{1}{2}\exp\left(\omega - \frac{1}{2}t\right) = -\frac{1}{2}y$

$$F_{\omega} = \exp\left(\omega - \tfrac{1}{2}t\right) = y$$
$$F_{\omega\omega} = y$$

$$\begin{aligned} dy &= \left(F_t + \tfrac{1}{2}F_{\omega\omega}\right)dt + F_\omega d\omega \\ &= \left(-\tfrac{1}{2}y + \tfrac{1}{2}y\right)dt + y\,d\omega \\ &= y\,d\omega \quad \checkmark \end{aligned}$$

(2) $\frac{dX}{X} = a\,dt + b\,d\omega \qquad \Rightarrow X = \exp\left(\left(a - \frac{b^2}{2}\right)t + b\omega\right)$

$$F_t = \left(a - \tfrac{b^2}{2}\right)X$$
$$F_\omega = bX$$
$$F_{\omega\omega} = b^2 X$$

$$\begin{aligned} dX &= \left(F_t + \tfrac{1}{2}F_{\omega\omega}\right)dt + F_\omega d\omega \\ &= \left(\left(a - \tfrac{b^2}{2}\right)X + \tfrac{1}{2}b^2X\right)dt + bX\,d\omega \\ &= aX\,dt + bX\,d\omega \end{aligned}$$

$$\Rightarrow \frac{dX}{X} = a\,dt + b\,d\omega \quad \checkmark$$

Geometric Brownian Motion

$X = e^{x}$ $\qquad F_X = F_{XX} = e^{x} = X$

$$\begin{aligned} dX &= \mu dt + \sigma dw \\ &= \left(\mu F_X + \tfrac{1}{2}\sigma^2 F_{XX}\right)dt + \sigma F_X dw \\ &= \left(\mu X + \tfrac{1}{2}\sigma^2 X\right)dt + \sigma X dw \end{aligned}$$

$$\Rightarrow \frac{dX}{X} = \left(\mu + \tfrac{1}{2}\sigma^2\right)dt + \sigma dw$$

3.22 Geometric Brownian Motion

$\frac{dX}{X} = \nu dt + \sigma dw$ diffusion process

$x = \log(X)$

Recall $y_t = f(x_t) \Rightarrow dy = \left(\mu f'(x) + \tfrac{1}{2}\sigma^2 f''(x)\right)dt + \sigma f'(x)dw$

So, $$\begin{aligned} dx &= \left(\frac{1}{X}\nu X + \tfrac{1}{2}(\sigma X)^2\left(-\frac{1}{X^2}\right)\right)dt + \sigma X \frac{1}{X}dw \\ &= \left(\nu - \tfrac{1}{2}\sigma^2\right)dt + \sigma dw \end{aligned}$$

Notice $\frac{dX}{X} \neq d\log(X)$ because $\log(\cdot)$ is a concave function.

Net Present Value

$$F(x_0) = E_0\left[\int_0^\infty f(x_t)e^{-\rho t}dt \,\middle|\, x(0) = x_0\right], \quad \rho > 0$$

$dx = \mu dt + \sigma dw$

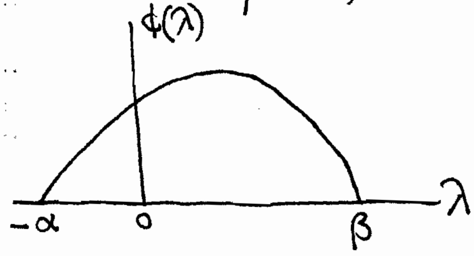
$f(\cdot)$ flow function

Example #1 : $f(x) = e^{\lambda x}$, $x \sim N(m, s^2)$ $\Rightarrow$ switch integral and expectations op.

Note $E[e^{\lambda x}] = \exp(\lambda m + \frac{1}{2}\lambda^2 s^2)$. $E[\lambda x_t] = \lambda(x + \mu t)$ $V[\lambda x_t] = \lambda^2 \sigma^2 t$

$$\Rightarrow E_0\left[\int_0^\infty e^{-\rho t} e^{\lambda x_t} dt \,\middle|\, x(0) = x_0\right] = \int_0^\infty e^{-\rho t} E_0\left[e^{\lambda x_t}\right] dt$$
$$= \int_0^\infty \exp\{\lambda(x+\mu t) + \tfrac{1}{2}\sigma^2\lambda^2 t - \rho t\} dt$$
$$= e^{\lambda x} \int_0^\infty \exp\{-(\rho - \lambda\mu - \tfrac{1}{2}\sigma^2\lambda^2) t\} dt$$
$$= e^{\lambda x} / (\rho - \lambda\mu - \tfrac{1}{2}\sigma^2\lambda^2)$$

For this to be positive, we need $\rho - \lambda\mu - \frac{1}{2}\sigma^2\lambda^2 > 0$. $\Leftrightarrow \lambda = \frac{\mu \pm \sqrt{\mu^2 + 2\rho\sigma^2}}{-\sigma^2} \doteq (-\alpha, \beta)$



We need $\lambda \in (-\alpha, \beta)$.

Example #2: ~~f(x)=~~ $f(X) = X^\lambda$ Geometric Brownian Motion : $\frac{dX}{X} = \nu dt + \sigma dw$

$x = \log(X)$

$$F(X) = E_0\left[\int_0^\infty e^{-\rho t} X_t^\lambda dt \,\middle|\, X(0) = X_0\right]$$
$$\Leftrightarrow F(x) = E_0\left[\int_0^\infty e^{-\rho t} e^{\lambda x_t} dt \,\middle|\, x_0 = \log(X_0)\right]$$
$$dx = (\nu - \tfrac{1}{2}\sigma^2) dt + \sigma dw$$

$\Rightarrow F(x) = X^\lambda / (\rho - \nu\lambda + \frac{1}{2}\sigma^2\lambda(1-\lambda)) > 0$ when $\rho - \nu\lambda + \frac{1}{2}\sigma^2\lambda(1-\lambda) > 0$

$\rho - \nu\lambda + \frac{1}{2}\sigma^2\lambda - \frac{1}{2}\sigma^2\lambda^2 = 0 \Rightarrow \lambda = \frac{(\frac{1}{2}\sigma^2 - \nu) \pm \sqrt{(\nu - \frac{1}{2}\sigma^2)^2 + 2\nu\sigma^2}}{-\sigma^2} = (-\gamma, \delta)$

If $\rho > \nu \Rightarrow \delta > 1$. We need $\lambda \in (-\gamma, \delta)$.

$$F(x) = E_0\left[\int_0^{\infty} e^{-\rho t} f(x_t)\,dt \,\middle|\, x(0) = x_0\right]$$
$$dx = \mu dt + \sigma dw$$

We want to approximate this in a Bellman equation fashion.

$$F(x) = f(x)\,dt + e^{-\rho dt} E\left[F(x+dx)\right] \qquad \text{(flow is assumed constant on } dt\text{)}$$

$$F(x) = f(x)dt + (1-\rho dt)\left\{F(x) + E\left[F(x+dx) - F(x)\right]\right\}$$
$$= f(x)dt + F(x) - \rho F(x)dt + E[dF]$$

$\Rightarrow \rho F(x)dt = f(x)dt + E[dF(x)]$ no arbitrage condition

normal rate of return = profit on $[0, dt]$ + Δvalue of asset

Using Ito's lemma, $E[dF] = \left[\mu F'(x) + \frac{1}{2}\sigma^2 F''(x)\right]dt$.

$\Rightarrow \rho F(x)dt = f(x)dt + \left[\mu F'(x) + \frac{1}{2}\sigma^2 F''(x)\right]dt$

$[*]$ $\rho F(x) = f(x) + \mu F'(x) + \frac{1}{2}\sigma^2 F''(x)$

$\Leftrightarrow \frac{1}{2}\sigma^2 F''(x) + \mu F'(x) + \rho F(x) + f(x) = 0$

linear differential equation with constant coefficients.

Homogeneous equation: $\frac{1}{2}\sigma^2 F''(x) + \mu F'(x) + \rho F(x) = 0$. Let $\hat{F}(x)$ be the solution of the homogeneous equation. The general solution is $F(x) = \hat{F}(x) + F^P(x)$, where $F^P(x)$ is the particular solution.

3.27 Ex. $f(x) = e^{\lambda x} \Rightarrow F(x) = Ae^{-\alpha x} + Be^{\beta x} + e^{\lambda x}$, $\lambda \in (-\alpha, \beta)$

Claim: $F(x+h) = e^{\lambda h} F(x)$. Define $y_t = x_t - h. \Rightarrow dy = \mu dt + \sigma dw$.

$e^{\lambda x_t} = e^{\lambda y} e^{\lambda h} \Rightarrow F(x+h) = e^{\lambda h} E\left[\int_0^{\infty} f(y_t) e^{-\rho t} \,\middle|\, y_0 = x_0\right] = e^{\lambda h} F(x)$.

$$\frac{F(x+h)-F(x)}{h} = \left(\frac{e^{\lambda h}-1}{h}\right)F(x) \Rightarrow F'(x) = \lambda F(x) \Rightarrow F(x) = Ke^{\lambda x} \quad (A=B=0)$$

for $\lambda \in (-\alpha, \beta)$.

Ex. Geometric Brownian Motion $G(x_0) = E_0\left[\int_0^\infty e^{-\varrho t} g(x_t)\,dt \mid x(0)=x_0\right]$

$$\frac{dX}{X} = \nu\,dt + \sigma\,dw$$

$\Rightarrow \varrho G(x_0)\,dt = g(x)\,dt + E[dG]$, $\quad E[dG] = G'(x)\nu X + \frac{1}{2}(\sigma X)^2 G''(x)$

$\Rightarrow \frac{1}{2}\sigma^2 X^2 G''(x) + \nu X G'(x) - \varrho G(x) + g(x) = 0$

Homogeneous equation: $\frac{1}{2}\sigma^2 X^2 G''(X) + \nu X G'(X) - \varrho G(X) = 0$

Try X^θ. $\Rightarrow \frac{1}{2}\sigma^2\theta(\theta-1)X^\theta + \theta\nu X^\theta - \varrho X^\theta = 0$

$\Rightarrow \frac{1}{2}\sigma^2\theta^2 + (\nu - \frac{1}{2}\sigma^2)\theta - \varrho = 0$. Roots $(-\gamma, \delta)$.

Convergence for $\theta \in (-\gamma, \delta)$.

$\Rightarrow \hat{G}(X) = c_1 X^{-\gamma} + c_2 X^\delta$

When $g(X) = X^\lambda$, $\lambda \in (-\gamma, \delta)$, the general solution is $G(X) = KX^\lambda$ $(c_1 = c_2 = 0)$.

$K = \left(\varrho - \nu\lambda - \frac{1}{2}\sigma^2\lambda(\lambda-1)\right)^{-1} > 0$ if $\lambda \in (-\gamma, \delta)$.

Example with control variable. By investing u, the mean shifts from μ to $\mu + u$ but at a cost of $c(u)$, $c' > 0$, $c'' \geq 0$.

$$\begin{aligned} V(x_0) &= \max_u E_0\left[\int_0^\infty e^{-\varrho t}\left(f(x_t) - c(u_t)\right)dt \mid x(0) = x_0\right] \\ &\cong \max_u \left\{\left(f(x_t) - c(u_t)\right)dt + e^{-\varrho t} E\left[V(x_0 + dx)\right]\right\} \\ &= \max_u \left\{\left(f(x_t) - c(u_t)\right)dt + (1 - \varrho\,dt)\left[V(x) + E[dV]\right]\right\} \end{aligned}$$

By Ito's lemma, $E[dV] = \left[(\mu + u)V'(x) + \frac{1}{2}\sigma^2 V''(x)\right]dt$.

Substituting this in and rearranging, we get
$$\rho V(x) = \max_u \left\{ f(x) - c(u) + (\mu + u)V'(x) + \tfrac{1}{2}\sigma^2 V''(x) \right\}$$

The first order condition is $V'(x) = c'(u^*)$, i.e., marginal value of a unit of investment equals its marginal cost.

Let $c(u) = \frac{1}{2}mu^2 \Rightarrow c'(u) = mu \Rightarrow u^* = \frac{V'(x)}{m}$.

$$\Rightarrow \rho V(x) = f(x) - \tfrac{1}{2}m\left(\frac{V'(x)}{m}\right)^2 + \left(\mu + \frac{V'(x)}{m}\right)V'(x) + \tfrac{1}{2}\sigma^2 V''(x)$$

General Case: $F(\tau, x) = \max_u E_\tau\left[\int_\tau^\infty f(x,u,t)\,dt \,\middle|\, x_\tau = x\right]$
s.t. $dx = \mu(x,u,t)\,dt + \sigma(x,u,t)\,dw$

$$F(t+\Delta t, x+\Delta x) = F(t,x) + F_t\Delta t + F_x \Delta x + \tfrac{1}{2}F_{xx}(\Delta x)^2 + \underset{\substack{\uparrow\\ \text{terms with } (\Delta t)^2 \text{ and higher}}}{o(\Delta t)}$$

Note $(\Delta x)^2 = \sigma^2 \Delta t$.
$$\Rightarrow F(t,x) = \max_u \left\{ f(x,u,t)\Delta t + E\left[F(t+\Delta t, x+\Delta x)\right]\right\} \quad \text{s.t. } \Delta x = \mu\Delta t + \sigma \Delta w$$
$$= \max_u \left\{ f(x,u,t)\Delta t + E\left[F(t,x) + F_t \Delta t + F_x(\mu\Delta t + \sigma\Delta w) + \tfrac{1}{2}\sigma^2 F_{xx}\Delta t + o(\Delta t)\right]\right\}$$
$E[\Delta w] = 0$ since $w \sim N(0,1)$.

$$\Rightarrow -F_t(t,x) = \max_u \left\{ f(x,u,t) + \mu F_x(t,x) + \tfrac{1}{2}\sigma^2 F_{xx}(t,x)\right\} \qquad \boxed{*}$$

Example: $F(0,x) = \min_u E\left[\int_0^\infty \{ e^{-rt}(ax^2 + bu^2)\,dt \,\middle|\, x_0 = x\right]$
$dx = u\,dt + \sigma x\,dw \quad , a>0, b>0, \sigma>0$

$$\Rightarrow -F_t = \min_u \left\{ e^{-rt}(ax^2 + bu^2) + uF_x + \tfrac{1}{2}\sigma^2 x^2 F_{xx}\right\}$$
FONC $[u]$: $2bu^* e^{-rt} = -F_x \quad \Rightarrow u^* = -\frac{e^{rt}F_x}{2b}$

Plugging in u^* and multiplying by e^{rt}, we have

$$-e^{rt}F_t = aX^2 + (4b)^{-1}e^{2rt}(F_X)^2 + \frac{1}{2}\sigma^2 X^2 e^{rt} F_{XX}$$

$$F(t,x) = e^{-rt}AX^2 \qquad \Rightarrow (A^2/b) + (r-\sigma^2)A - a = 0$$

$$\Rightarrow A = \frac{b}{2}\left[(\sigma^2 - r) \pm \sqrt{(r-\sigma^2)^2 + 4\frac{a}{b}}\right] \qquad \Rightarrow u^* = -\frac{Ax}{b}$$

3.29 Time-autonomous problems: $\mu = \mu(x,u)$, $\sigma = \sigma(x,u)$, $\hat{f}(x,u,t) = g(t)f(x,u)$

$$V(x) = \max_u E_0\left[\int_0^\infty e^{-rt} f(x,u)\,dt \,\middle|\, x(0) = x\right]$$

$$\text{s.t. } dx = \mu(x,u)\,dt + \sigma(x,u)\,dw$$

$$\Rightarrow rV(x) = \max_u \left\{ f(x,u) + \mu(x,u)V'(x) + \frac{1}{2}(\sigma(x,u))^2 V''(x) \right\}$$

Example: Consumption/Investment over time.

- Two assets: safe asset with rate of return s and risky asset with expected return a and standard deviation σ. $a > s$
- Initial wealth $\omega(0) = \omega > 0$
- Consumption c_t
- θ_t is the fraction of wealth invested in the risky asset
- Preferences exhibit constant relative risk aversion: $u(c_t) = \frac{c_t^b}{b}$, where $(1-b)$ is the coefficient of relative risk aversion and $b \in (0,1)$.

$$V(\omega) = \max_{c,\theta} E_0\left[\int_0^\infty e^{-rt} \frac{c_t^b}{b}\,dt \,\middle|\, \omega(0) = \omega\right]$$

$$\text{s.t. } d\omega = [(1-\theta)s\omega + \theta a\omega - c]\,dt + (\theta\omega\sigma)\,dz \quad , z \sim N(0,1)$$

Since this is a time autonomous problem,

$$rV(\omega) = \max_{c,\theta} \left\{ \frac{c^b}{b} + \left((1-\theta)s\omega + \theta a\omega - c\right)V'(\omega) + \frac{1}{2}\theta^2\omega^2\sigma^2 V''(\omega) \right\}$$

FONC $[c]$: $c^{b-1} = V'(\omega) \Rightarrow c^* = \left[V'(\omega)\right]^{\frac{1}{b-1}}$

$[\theta]$: $(a-s)\omega V'(\omega) + \theta\omega^2\sigma^2 V''(\omega) = 0 \Rightarrow \theta^* = \frac{(a-s)}{\sigma^2} \cdot \left(\frac{-V''(\omega)\cdot\omega}{V'(\omega)}\right)^{-1}$

Coefficient of relative risk aversion

$$rV(\omega) = \frac{1}{b}\left[V'(\omega)\right]^{\frac{b}{b-1}} + s\omega V'(\omega) + (a-s)\omega V'(\omega)\frac{(a-s)}{\sigma^2}\left(\frac{-V''(\omega)\omega}{V'(\omega)}\right)^{-1}$$
$$+ \frac{1}{2}\omega^2\sigma^2 V''(\omega)\left[\frac{(a-s)}{\sigma^2}\left(\frac{-V''(\omega)\omega}{V'(\omega)}\right)^{-1}\right]^2$$

Try $V(\omega) = A\omega^b$. $V'(\omega) = bA\omega^{b-1}$, $V''(\omega) = b(b-1)A\omega^{b-2}$ $\quad \frac{-V''\omega}{V'} = 1-b$

$$rA\omega^b = \frac{1}{b}\left(Ab\omega^{b-1}\right)^{\frac{b}{b-1}} + s\omega Ab\omega^{b-1} + \frac{(a-s)^2\omega}{\sigma^2}Ab\omega^{b-1}(1-b)^{-1}$$

$$\Rightarrow rA = \frac{1}{b}(Ab)^{\frac{b}{b-1}} + sAb + \frac{(a-s)^2}{\sigma^2}\frac{Ab}{1-b}$$

$$\Rightarrow A\left(r - sb - \frac{(a-s)^2}{\sigma^2}\frac{b}{1-b}\right) = A^{\frac{b}{b-1}} \cdot b^{\frac{-b+1}{b-1}} \cdot b^{\frac{b}{b-1}} = A^{\frac{b}{b-1}} b^{\frac{1}{b-1}}$$

$$\Rightarrow A^{\frac{-1}{b-1}}(\cdot) = b^{\frac{1}{b-1}}$$

$$\Rightarrow bA = \left(r - sb - \frac{(a-s)^2}{\sigma^2}\frac{b}{1-b}\right)^{b-1} \Leftrightarrow A = \frac{1}{b}\left(r - sb - \frac{(a-s)^2}{\sigma^2}\frac{b}{1-b}\right)^{b-1}$$

$c_t^* = \left[V'(\omega)\right]^{\frac{1}{b-1}} = \left(Ab\omega_t^{b-1}\right)^{\frac{1}{b-1}} = \left(r - sb - \frac{(a-s)^2}{\sigma^2}\frac{b}{1-b}\right)\omega_t$ *linear in wealth*

$\theta_t^* = \frac{(a-s)}{(1-b)\sigma^2}$ *constant*

Comparative Statics

$$\theta_t^* = \frac{(a-s)}{(1-b)\sigma^2} \qquad \frac{d\theta^*}{d(a-s)} = \frac{1}{(1-b)\sigma^2} > 0 \qquad \frac{d\theta^*}{d(1-b)} = -\frac{(a-s)}{(1-b)^2\sigma^2} < 0 \qquad \frac{d\theta^*}{d\sigma^2} = -\frac{(a-s)}{(1-b)\sigma^4} < 0$$

An increase in the expected premium from the risky asset leads to an increase in the portfolio holdings in the risky asset. An increase in relative risk aversion or the variance of the risky asset decreases the portfolio holdings in the risky asset.

$$c_t^* = \left(r - sb - \frac{(a-s)^2}{\sigma^2}\frac{b}{1-b}\right)\omega_t$$

$$\frac{dc_t^*}{d\sigma^2} = \frac{(a-s)^2}{\sigma^4}\frac{b}{1-b}\omega_t > 0$$ $\Rightarrow$ savings declines as variance of risky asset increases.

$$\frac{dc_t^*}{db} = \left(-s - \frac{(a-s)^2}{\sigma^2}\frac{1}{(1-b)^2}\right)\omega_t = -\left(s + \frac{(a-s)^2}{\sigma^2}\frac{1}{(1-b)^2}\right)\omega_t < 0$$

A decrease in relative risk aversion leads to a decrease in consumption (increase in savings).

$$\beta V(K_{t+1}) = \max_{0 \leq c_{t+1} \leq K_{t+1}} \left\{ \beta \frac{c_{t+1}^{1-\alpha}}{1-\alpha} + \beta\delta E_{t+1}\left[V(K_{t+2})\right] \right\}$$

$$W(K_{t+1}) = \max_{0 \leq c_{t+1} \leq K_{t+1}} \left\{ \frac{c_{t+1}^{1-\alpha}}{1-\alpha} + \beta\delta E_t\left[V(K_{t+2})\right] \right\}$$

$$\beta V(K_{t+1}) = W(K_{t+1}) + \left(\overset{\beta-1}{\cancel{1-\beta}}\right) \frac{c_{t+1}^{1-\alpha}}{1-\alpha} = W(K_{t+1}) + (1-\beta)\, \frac{\left(g(K_{t+1})\right)^{1-\alpha}}{1-\alpha}$$

$$\beta V'(K_{t+1}) = W'(K_{t+1}) + \left(\overset{\beta-1}{\cancel{1-\beta}}\right)\left(g(K_{t+1})\right)^{-\alpha} g'(K_{t+1})$$

$$\left(g(K_t)\right)^{-\alpha} \geq \delta E_t\left[\tilde{r}_t\, \beta V'(K_{t+1})\right] = \delta E_t\left[\left(g(K_{t+1})\right)^{-\alpha} + \left(\overset{\beta-1}{\cancel{1-\beta}}\right) g'(K_{t+1}) \left(g(K_{t+1})\right)^{-\alpha}\right]$$

$$= \delta E_t\left[\left(1 + \left(\overset{\beta-1}{\cancel{1-\beta}}\right) g'(K_{t+1})\right)\left(g(K_{t+1})\right)^{-\alpha}\right]$$

$$= E_t\left[\tilde{r}_t\left(\beta\delta g'(K_{t+1}) + \delta\left(1 - g'(K_{t+1})\right)\right)\left(g(K_{t+1})^{-\alpha}\right)\right]$$

$$\beta = 1 \Rightarrow \left(g(K_t)\right)^{-\alpha} \geq E_t\left[\tilde{r}_t\, \delta\, \cancel{g'(K_{t+1})} \left(g(K_{t+1})\right)^{-\alpha}\right]$$

$$= \text{ for } g(K_t) < K_t$$

Posit $g(K_t) = \lambda K_t \Rightarrow g'(K_t) = \lambda$ $\qquad K_{t+1} = r_t(K_t - \lambda K_t)$

$$= \tilde{r}_t (1-\lambda) K_t$$

$$\cancel{(\lambda K_t)^{-\alpha}} \geq E_t\left[\tilde{r}_t\left(\beta\delta\lambda + \delta(1-\lambda)\right)\left(\cancel{\lambda}\, \tilde{r}_t (1-\lambda) \cancel{K_t}\right)^{-\alpha}\right]$$

$$1 \geq E_t\left[\tilde{r}_t\, \delta(\beta\lambda + 1 - \lambda)\, \tilde{r}_t^{-\alpha} (1-\lambda)^{-\alpha}\right]$$

$$(1-\lambda)^{\alpha} \geq E_t\left[\tilde{r}_t^{1-\alpha}\left(1 - (1-\beta)\lambda\right)\right]$$

$$-\alpha (1-\lambda)^{\alpha-1} d\lambda = E_t\left[\tilde{r}_t^{1-\alpha}(\beta-1) d\lambda + \tilde{r}_t^{1-\alpha} \lambda\, d\beta\right] = E_t\left[\tilde{r}_t^{1-\alpha}\left(\lambda\, d\beta - (1-\beta)\, d\lambda\right)\right]$$

(2) $$\max_{\{c_t\}_{t=0}^{\infty}} E_0\left[\frac{c_0^{1-\alpha}}{1-\alpha}+\beta\sum_{t=1}^{\infty}\delta^t\frac{c_t^{1-\alpha}}{1-\alpha}\right] \quad , \alpha>0,\ \beta>0,\ 0<\delta<1$$

$$\text{s.t.}\quad K_{t+1}=\tilde{r}_t\left[K_t-c_t\right],\ K_0>0 \text{ given}$$

$$\{\tilde{r}_t\}_{t=0}^{\infty} \text{ iid}$$

(a) Current Value Function

$$W(K_0)=\max_{0\leq c_0\leq K_0}\left\{\frac{c_0^{1-\alpha}}{1-\alpha}+\beta\delta\, E_0\left[V\left(\tilde{r}_0(K_0-c_0)\right)\right]\right\}$$

$$V(K_1)=u(c_1)+\delta E_1\left[V\left(\tilde{r}_1(K_1-c_1)\right)\right]$$ Continuation Value Function

$$\Leftrightarrow W(K_t)=\max_{0\leq c_t\leq K_t}\left\{\frac{c_t^{1-\alpha}}{1-\alpha}+\beta\delta E_t\left[V\left(\underbrace{\tilde{r}_t(K_t-c_t)}_{K_{t+1}}\right)\right]\right\} \qquad [1]$$

$$V(K_{t+1})=\max_{0\leq c_{t+1}\leq K_{t+1}}\left\{\frac{c_{t+1}^{1-\alpha}}{1-\alpha}+\delta E_{t+1}\left[V\left(\tilde{r}_{t+1}(K_{t+1}-c_{t+1})\right)\right]\right\} \qquad [2]$$

[1]'s FONC $[c_t]$: $$c_t^{-\alpha}\geq\beta\delta E_t\left[\tilde{r}_t V'\left(\tilde{r}_t(K_t-c_t)\right)\right]$$
$$= \text{ if } c_t<K_t$$

$g(K)$ optimal consumption function

$$\left(g(K_t)\right)^{-\alpha}\geq\beta\delta E_t\left[\tilde{r}_t V'\left(\tilde{r}_t\left(K_t-g(K_t)\right)\right)\right]$$
$$= \text{ if } g(K_t)<K_t$$

Envelope theorem + FOC $\Rightarrow W'(K_t)=u'(g(K_t))=\left(g(K_t)\right)^{-\alpha}$

$$\left[(1-\beta)E_t\left[\tilde{r}_t^{1-\alpha}\right]-\alpha(1-\lambda)^{\alpha-1}\right]d\lambda=\lambda E_t\left[\tilde{r}_t^{1-\alpha}\right]d\beta$$

$$\frac{d\lambda}{d\beta}=\frac{E_t\left[\tilde{r}_t^{1-\alpha}\right]}{\left((1-\beta)E_t\left[\tilde{r}_t^{1-\alpha}\right]-\alpha(1-\lambda)^{\alpha-1}\right)}$$

$$\alpha=1 \Rightarrow \quad \frac{d\lambda}{d\beta}=\frac{1}{(1-\beta)\cdot 1-1}=-\frac{1}{\beta}<0$$

$$1-\lambda \geqslant \left(E_t\left[\tilde{r}_t^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}$$

$$\lambda \leqslant 1-\left(E_t\left[\tilde{r}_t^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}$$

$$V(x) = \max_u E\left[\int_0^\infty f(x,u)e^{-rt}dt \,\Big|\, x_0 = x\right]$$
$$\text{s.t. } dx = \mu(x,u)dt + \sigma(x,u)dw$$

$$\Rightarrow rV(x) = \max_u E\left[f(x,u) + \mu(x,u)V'(x) + \tfrac{1}{2}\sigma^2(x,u)V''(x)\right]$$

Example: portfolio choice and consumption over time.

control variables: $c_t, \theta_t \leftarrow$ in risky asset $\leftarrow$ expected return a, var. σ^2 $(1-\theta_t)$ in riskless asset w/return s

$$V(\omega) = \max_{c,\theta} E\left[\int_0^\infty e^{-rt}\frac{c_t^b}{b}dt \,\Big|\, \omega_0 = \omega\right] \quad , b \leq 1 \quad \Rightarrow V(\cdot) \text{ concave in } c$$

$$\text{s.t. } d\omega = \left[(1-\theta)\omega s + \theta\omega a - c\right]dt + (\theta\omega)\sigma d\bar{\omega} \quad , a > s$$

This is a time autonomous ~~equation~~ problem.

$$rV(\omega) = \max_{c,\theta} \left[\frac{c_t^b}{b} + \left((1-\theta)\omega s + \theta\omega a - c\right)V'(\omega) + \tfrac{1}{2}\theta^2\omega^2\sigma^2 V''(\omega)\right]$$

$$\text{FONC } [c]: \; c_t^{b-1} = V'(\omega) \quad \Rightarrow c_t^* = \left[V'(\omega)\right]^{\frac{1}{b-1}}$$

$$[\theta]: \; (-\omega s + \omega a)V'(\omega) + \theta\omega^2\sigma^2 V''(\omega) = 0$$

$$\Rightarrow \theta_t^* = \frac{(s-a)\omega V'(\omega)}{\omega^2\sigma^2 V''(\omega)} = \frac{-(a-s)}{\omega\sigma^2}\frac{V'(\omega)}{V''(\omega)}$$

$$rV(\omega) = \frac{1-b}{b}\left[V'(\omega)\right]^{\frac{b}{b-1}} + \omega s + \omega(a-s)\left[\frac{-(a-s)}{\omega\sigma^2}\frac{V'(\omega)}{V''(\omega)}\right] + \tfrac{1}{2}\omega^2\sigma^2\left[\frac{-(a-s)}{\omega\sigma^2}\frac{V'(\omega)}{V''(\omega)}\right]^2 V''(\omega)$$

$$rV(\omega) = \frac{1-b}{b}\left[V'(\omega)\right]^{\frac{b}{b-1}} + sWV'(\omega) - \frac{(a-s)^2(V'(\omega))^2}{2\sigma^2 V''(\omega)}$$

$V(\omega) = AW^b$?

$$rAW^b = \frac{1-b}{b}\left[bAW^{b-1}\right]^{\frac{b}{b-1}} + sWbAW^{b-1} - \frac{(a-s)^2}{2\sigma^2}\frac{(bAW^{b-1})^2}{b(b-1)AW^{b-2}}$$

$$rA = \frac{1-b}{b} + sbA - \frac{(a-s)^2 bA}{2\sigma^2(b-1)}$$

$$Ab = \left\{\left[r - sb - \frac{(a-s)^2 b}{2\sigma^2(1-b)}\right]\frac{1}{1-b}\right\}^{b-1} \Rightarrow A = \frac{1}{b}\left\{\cdot\right\}^{b-1}$$

#2

$$c_t^* = \left[V'(\omega)\right]^{\frac{1}{b-1}} = \left[Ab\right]^{\frac{1}{b-1}} W_t$$ linear function of wealth

$\uparrow$ constant

$$\theta^* = \frac{a-s}{(1-b)\sigma^2}$$

$\cancel{\frac{d(a-s)}{d\theta} > 0}$ $\quad \frac{d\theta^*}{d(a-s)} > 0 \quad \frac{d\theta^*}{d\sigma^2} = -\frac{(a-s)}{(1-b)(\sigma^2)^2} < 0$

$$\frac{d\theta^*}{d(1-b)} = -\frac{(a-s)}{(1-b)^2\sigma^2} < 0$$

more risk averse
$\Rightarrow$ less optimally in ~~rsk~~ risky asset

$\sigma^2 \uparrow \Rightarrow A \downarrow$

$$\frac{dc^*}{d\sigma^2} \; \cancel{\text{because } A \text{ increases}} = \frac{-1}{1-b} A^{\frac{-b}{b-1}} b^{\frac{1}{b-1}} W \frac{dA}{d\sigma^2} > 0$$

~~higher~~ lower savings

Rothschild + Stiglitz uncertainty in future income (not rate of return)

$\uparrow$ savings when more uncertainty when $u''' > 0$

$$c^* = \left\{\left(r - sb - \frac{(a-s)^2 b}{2\sigma^2(1-b)}\right)\frac{1}{1-b}\right\}\omega$$

$$\frac{dc^*}{db} = ?$$

$b < \frac{1}{3} \Rightarrow c\downarrow$ as $b\downarrow$

$b \in \left[\frac{1}{3}, 1\right]$, it depends

April 3, 5, 10, 12
start at 12:30, end 2:10
Cancel April 17 meeting
Exam on April 12, due April 13 5pm

Firm Investment Under Uncertainty (Abel, AER, 1983)

- Price uncertainty in future
- Competitive firm using capital and labor.
- Price of capital fixed
- Output: $L^{\alpha}K^{1-\alpha}$, $0<\alpha<1$

  fixed wage rate ω (Z will denote Wiener process)
- Capital depreciates at rate $\delta>0$
- Cost of investment: $C(I)$ $C'>0, C''>0$

  $=\gamma I^{\beta}$, $\beta>1$, $\gamma>0$
- $dK_t = (I_t - \delta K_t)dt$
- random price P_t: $\dfrac{dP_t}{P_t} = \sigma dz$, $z \sim N(0,1)$

  $E_t[P_s] = P_t$ $V_t[P_s] = \sigma^2(s-t)$ $s>t$
- Risk neutral firm

$$V(K_t,P_t) = \max_{I_s,L_s} E_t\left[\int_0^{\infty}\left(P_s L_s^{\alpha} K_s^{1-\alpha} - \omega_s L_s - \gamma I_s^{\beta}\right)e^{-r(s-t)}\,ds\right]$$

$$\text{s.t. } \frac{dP_t}{P_t} = \sigma dz$$

$$V(K_t,P_t) = \max_{I_s,L_s}\left\{\left(P_t L_t^{\alpha} K_t^{1-\alpha} - \omega_t L_t - \gamma I_t^{\beta}\right)ds + \overset{e^{-\rho dt}}{\widehat{(1-\rho\,dt)}}\, E\left[V(K+dK, P+dP)\right]\right\}$$

$$= \max_{I_s,L_s}\left\{\left(PL^{\alpha}K^{1-\alpha} - \omega L - \gamma I^{\beta}\right)ds + (1-\rho ds)\left[V(K_t,P_t) + E[dV]\right]\right\}$$

$$\rho V(K_t,P_t)ds = \max_{I_s,L_s}\left\{\left(P_s L_s^{\alpha} K_s^{1-\alpha} - \omega_s L_s - \gamma I_s^{\beta}\right)ds + E[dV]\right\}$$

4.3 #1

$$rV(K_t,P_t)dt = \max_{L_t,I_t}\left\{\left(P_tL_t^{\alpha}K_t^{1-\alpha}-\omega_t L_t-\gamma I_t^{\beta}\right)dt + E_t[dV]\right\}$$

$$dV = V_K dK + V_p dp + \tfrac{1}{2}V_{KK}(dK)^2 + \tfrac{1}{2}V_{PP}(dP)^2 + V_{PK}\,dK\,dP$$

$$dK_t = [I_t - \delta K_t]dt$$

$dP_t = P_t\sigma dz$, z Wiener Process $N(0,1)$

$\Rightarrow E[dP_t] = 0 \quad \to V_p dP \to 0$

	dz	dt
dz	dt	0
dt	0	0

$(dK)^2$ has $(dt)^2 \approx 0 \quad \Rightarrow \tfrac{1}{2}V_{KK}(dK)^2 \to 0$

$(dK)(dP) \approx dt\cdot dt \approx 0 \quad V_{PK}\,dK\,dP \to 0$

$(dP)^2 = P_t^2\sigma^2(dz)^2 = P_t^2\sigma^2 dt$

$\Rightarrow E[dV] = V_K(I_t-\delta K_t)dt + \tfrac{1}{2}V_{PP}P_t^2\sigma^2 dt$

$$\Rightarrow rV(K_t,P_t) = \max_{L_t,I_t}\left\{P_tL_t^{\alpha}K_t^{1-\alpha}-\omega_t L_t-\gamma I_t^{\beta} + V_K(K_t,P_t)(I_t-\delta K_t)+\tfrac{1}{2}V_{PP}(K_t,P_t)P_t^2\sigma^2\right\}$$

Take FOC's wrt I_t, L_t. ~~Unsure~~ Solve for I_t^* and L_t^*. Substitute back into the above expression.

$[L_t]: \alpha P_t L_t^{\alpha-1}K_t^{1-\alpha} = \omega$

$[I_t]: \beta\gamma I_t^{\beta-1} = V_K(K_t,P_t)$

Then solve the nasty PDE.

$$V(K_t,P_t) = g_t K_t + M_t P_t^{\frac{\beta}{(\beta-1)(1-\alpha)}}$$

I_t is monotone function of g_t.

$\sigma^2\uparrow \Rightarrow I_t\uparrow$. Investment increases with more volatility in price.

$$F(x) = E_0\left[\int_0^\infty e^{-\rho t} f(x_t)\,dt \,\Big|\, x_0 = x\right]$$

$$dx = \mu dt + \sigma dw$$

$$F(x) = \frac{e^{\lambda x}}{\rho - \mu\lambda - \frac{1}{2}\sigma^2\lambda^2}$$ when $f(\cdot)$ exponential $f(x_t) = e^{\lambda x_t}$

$$E\left[\int_0^\infty x_t^n e^{-\rho t}\,dt \,\Big|\, x_0 = x\right] \quad , n = 1, 2, 3, \dots$$

$$e^{\lambda x} = \sum_{n=1}^{\infty} \frac{(\lambda x)^n}{n!}$$

$$E\left[\int_0^\infty e^{-\rho t} \sum_{n=1}^{\infty} \frac{(\lambda x_t)^n}{n!}\,dt \,\Big|\, x_0 = x\right]$$

$$= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} E\left[\int_0^\infty x_t^n e^{-\rho t} \,\Big|\, x_0 = x\right]$$

$$e^{\lambda x_t} = \sum_{n=1}^{\infty} \frac{\lambda^n x_t^n}{n!} \qquad \frac{1}{\rho} \frac{1}{1 - \left[\frac{\lambda\mu}{\rho} + \frac{0.5\sigma^2\lambda^2}{\rho}\right]} = \frac{1}{\rho}\sum_{n=1}^{\infty} (\ )^n$$

Coefficients of same order of λ must be same

$$E\left[\int_0^\infty x_t e^{-\rho t}\,dt\right] = \frac{\mu}{\sigma^2} + \frac{x}{\rho}$$

$$E\left[\int_0^\infty e^{-\rho t} x_t^2\,dt\right] = \left(\frac{\sigma^2}{\rho^2} + \frac{2\mu^2}{\rho^3}\right) + \frac{2\mu x}{\rho^2} + \frac{x^2}{\rho}$$

Non-constant discount factor

From experiments

\$100 now vs. \$110 a month from now

Many take \$100.

\$100 two years from now vs. \$110 two years + one month from now

Many take \$110.

Dynamic inconsistency.

Suppose every month that passes reduces chance of payment by 1%.

$(100, 1)$ vs $(110, 0.99)$
$\uparrow$

$(100, 0.7857)$ vs $(110, 0.7778)$
$\uparrow$

When probabilities close to 0, 1, people don't maximize expected utility

Laibson: current discounting heavier than future discounting

(β, δ) method (quasi-hyperbolic discounting)

- one liquid asset and one illiquid asset. (x_t, z_t)
  (must make decision one period ahead)
- $c_t \leq y_t + R x_{t-1}$ [1]
  $\uparrow$ income in period t
- $x_t + z_t = y_t + R(x_{t-1} + z_{t-1}) - c_t$ [2]

~~max~~ $\cancel{E_t\left[\sum_{\tau=t}\right.}$

$$\max_{x_\tau, z_\tau} E_t\left[u(c_t) + \beta \sum_{\tau=t}^{T} \delta^{\tau-t} u(c_\tau)\right]$$

$\beta \in (0,1)$

s.t. [1], [2]

$x_t \geq 0, \; z_t \geq 0, \; c_t \geq 0 \; \forall t$

$$MRS^{t}_{t+1,t+2} = \frac{u'(c_{t+1})}{\delta u'(c_{t+2})}$$

$$MRS^{t+1}_{t+1,t+2} = \frac{u'(c_{t+1})}{\beta\delta u'(c_{t+2})}$$

$$MRS^{t}_{t+1,t+2} < MRS^{t+1}_{t+1,t+2}$$

$f(\tau)$ rate of discounting $\Rightarrow -\frac{f'(\tau)}{f(\tau)}$

exponential: $f(\tau) = e^{-\delta t} \Rightarrow -\frac{f'(\tau)}{f(\tau)} = \delta$ consant

hyperbolic: $f(\tau) = (1+\gamma\tau)^{-\gamma/\alpha} \Rightarrow$ rate $= \frac{\gamma}{1+\alpha\tau}$

quasi-hyperbolic $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \dots\}$

Can we find equilibrium? What can we say about it?

Assumption: $u'(y_t) \geq \max_{\tau} \beta\delta^{\tau}R^{\tau}u'(y_{t+\tau})$ for all $t, \tau \geq 0$

$u(\cdot)$ constant relative risk aversion: $u(c) = \frac{c^{1-\rho}}{1-\rho}$, $\rho > 0$

$\rho = 1 \Rightarrow \frac{1}{y_t} \geq \beta\delta R \frac{1}{y_{t+1}} \Leftrightarrow \frac{y_{t+1}}{y_t} \geq \beta\delta R$

unique subgame perfect Nash equilibrium

Necessary and sufficient conditions

(a) $u'(c_t) \geq \max_{\tau} \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau})$, $t, \tau \geq 0$

(b) $u'(c_t) > \max_{\tau} (\cdot) \Rightarrow c_t = y_t + Rx_{t-1}$ (consume all liquid assets)

(c) $u'(c_{t+1}) < \max_{\tau} \delta^{\tau}R^{\tau}u'(c_{t+1+\tau}) \Rightarrow x_t = 0 \quad \forall t, \tau \geq 0$

(d) $u'(c_{t+1}) > \max_{\tau} \delta^{\tau}R^{\tau}u'(c_{t+1+\tau}) \Rightarrow z_t = 0 \quad \forall t, \tau \geq 0$

- $u'(c_t) \geq \max_{\tau} \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau})$ [a] • $u'(y_t) \geq \max_{\tau \geq 1} \beta\delta^{\tau}R^{\tau}u'(y_{t+\tau})$ $\forall t, \tau > 0$

  $> \Rightarrow c_t = y_t + Rx_{t-1}$ [b]

$c_t = c_t(x_{t-1}, z_{t-1})$

Claim: $\left.\frac{\partial c_t}{\partial x_{t-1}}\right|_{equil.} = 1$ $\left.\frac{\partial c_t}{\partial z_{t-1}}\right|_{equil.} = 0$

Derivation of this depends on uniqueness of the equilibrium

In equilibrium, $c_t = y_t + Rx_{t-1}$ for $t \geq 2$.

Assume strict inequality in [a] does not hold for some t, τ, $t \geq 2$.

$\Rightarrow u'(c_t) = \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau})$

$u'(c_t) < \delta^{\tau}R^{\tau}u'(c_{t+\tau}) \Rightarrow x_{t-1} = 0 \Rightarrow c_t = y_t$

$\Rightarrow u'(y_t) = \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau}) \leq \beta\delta^{\tau}R^{\tau}u'(y_{t+\tau}) \Rightarrow\Leftarrow$

$\Rightarrow$ must have strict inequalities

$u'(c_t) > \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau})$ $\forall t \geq 2, \tau \geq 0$

For some $\varepsilon > 0$ small

$u'(c_t + \varepsilon) > \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau})$ $\forall t \geq 2, \tau \geq 0$

$\Rightarrow c_t =$ all liquid assets $\left(\left.\frac{\partial c_t}{\partial x_{t-1}}\right|_{eq.} = 1\right)$

$u'(c_t) > \beta\delta^{\tau}R^{\tau}u'(c_{t+\tau} + R^{\tau}\cdot\varepsilon)$ for some $\varepsilon > 0$ small

$\Rightarrow c_t =$ all liquid assets $\left(\left.\frac{\partial c_t}{\partial z_{t-1}}\right|_{eq.} = 0\right)$

Using (β,δ) model, what can we do with value functions?

Harris-Laibson (Econometrica, 2001)

- one asset x_t
- $0 \leq c_t \leq x_t$ (no borrowing) [1]
- $\{\tilde{y}_t\}$ iid random income with density $f(y)$
- $x_{t+1} = R(x_t - c_t) + \tilde{y}_{t+1}$ [2]

$$V(x_0) = \max_{\{c_t\}} E_0\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right]$$

Exponential Discounting

$$\text{s.t. } 0 \leq c_t \leq x_t$$
$$x_{t+1} = R(x_t - c_t) + \tilde{y}_{t+1}$$

Euler equations: $u'(c_t^*) \geq \delta E_t\left[R u'(c_{t+1}^*)\right] \quad \forall t \geq 0$

$=$ if $c_t^* < x_t$

What happens with (β,δ) model?

Consider (β,δ) model with $\beta<1$. What is an eq. here?
It is a pair of value functions $W(x), V(x)$ and an optimal decision rule for consumption $c(x)$ such that

(a) given the continuation value function $V(x)$, the current value function $W(x)$ satisfies: $W(x) = \max_{0 \leq c \leq x} \{u(c) + \beta\delta E[V(R(x-c)+\tilde{y})]\}$
and the max is given by $c(x) = c^*$.

(b) given $c(x)$, $V(x)$ satisfies $V(x) = u(c(x)) + \delta E\left[V(R(x-c(x))+\tilde{y}\right]$

4.5 Along eq. path of $\{x_t\}$,

$$W(x_t) = u(c(x_t)) + \beta\delta E_t\left[V(R(x_t - c(x_t)) + \tilde{y}_{t+1}\right]$$

$$V(x_{t+1}) = u(c(x_{t+1})) + \delta E_{t+1}\left[V(x_{t+2})\right]$$

$$W'(x_t) = u'(c(x_t)) \quad \forall x_t$$

$$u'(c(x_t)) - \beta\delta E_t\left[RV'(R(x_t - c(x_t)) + \tilde{y}_{t+1})\right] \geq 0$$ ~~Bellman eq.~~ Euler eq.

$$= 0 \text{ if } c(x_t) < x_t$$

$$W(x_{t+1}) = u(c(x_{t+1})) = \beta\delta E_{t+1}\left[V(R(x_{t+1} - c(x_{t+1})) + \tilde{y}_{t+1})\right]$$

$$\Rightarrow \beta V(x_{t+1}) = \beta u(c(x_{t+1})) + W(x_{t+1}) - u(c(x_{t+1}))$$

$$= W(x_{t+1}) - (1-\beta)u(c(x_{t+1}))$$

$$\Rightarrow \beta V'(x_{t+1}) = W'(x_{t+1}) - (1-\beta)u'(c(x_{t+1}))\cdot c'(x_{t+1})$$ ~~$c(x_t) < x_t$~~

$$= u'(c(x_{t+1})) - (1-\beta)u'(c(x_{t+1}))\cdot c'(x_{t+1})$$

Euler ~~Bellman~~ eq. becomes $u'(c(x_t)) \geq \cancel{\beta}\, \delta E_t\left[R(u'(c(x_{t+1})))(1-(1-\beta)c'(x_{t+1}))\right]$

$=$ for interior solution

$$\geq E_t\left[Ru'(c(x_{t+1}))\left[\beta\delta c'(x_{t+1}) + (1-c'(x_{t+1}))\delta\right]\right]$$

strong hyperbolic Euler eq.

Effective discount factor : $\beta\delta c'(x_{t+1}) + \delta(1-c'(x_{t+1}))$

stochastic, weighted avg. of δ and $\beta\delta$, time dependent

$c' \to 1 \Rightarrow \beta\delta$ is discount rate

$c' \to 0 \Rightarrow \delta$ is discount rate

4.5 #4

Conditions that guarantee existence:

- u is CRRA, $u(c) = \dfrac{c^{1-\rho}}{1-\rho}$, $\rho > 0$, $0 < \underline{\rho} \leq \rho \leq \bar{\rho} < \infty$
- $f(y)$ satisfies for some $0 < \underline{y} < \bar{y} < \infty$, $f(y) = 0$ outside $[\underline{y}, \bar{y}]$
- $\cancel{\min}$ $\max \{\delta, \delta R^{1-\underline{\rho}}\} < 1$

Example: Krusell, Kuruscu, Smith JET, 2002

Quasi-hyperbolic Discounting

- Two sources of income: assets & labor
- $u = \log(c)$, $\delta \in (0,1)$, $\beta > 0$
- $c + k' = Ak^{\alpha}$ $\quad k' = k_{t+1}$, full depreciation
- $r = \alpha A k^{\alpha - 1}$
- $\omega = (1-\alpha) A k^{\alpha}$

Current self perceives $k_{t+1} = g(k_t)$

Current self solves $V_0(k) = \max\limits_{k'} u(rk + \omega - k') + \beta \delta V(k')$

$$V(k) = u(rk + \omega - g(k)) + \delta V(g(k))$$

Eq. is (V_0, V, g).

* Optimal Pigouvian tax in face of (β, δ) consumer?

In eq. $V(k) = a + b \log\left(k + \frac{\omega}{r-1}\right)$ $\quad$ $k + \frac{\omega}{r-1}$ proportional to $rk + \omega \sum_{i=0}^{\infty} \frac{1}{r^i}$

$$g(k) = s\left(rk + \frac{r}{r-1}\omega\right), \quad s = \frac{\beta\delta}{1-\delta(1-\beta)}$$

$$W' = srW$$

Krusell, Kuruşçu, and Smith (2001)

Competitive eq. : price-taking behavior, consumers + producers maximize

k - individual's holdings of capital

$\bar{k}$ - per capita capital stock.

$g(k,\bar{k})$

$V(k,\bar{k})$ - continuation value function

$\bar{k}' = G(\bar{k})$ law of motion for per capita capital stock

$(r(\bar{k}), \omega(\bar{k}))$

(1) $\max_{k'} \left\{ \log\left(r(\bar{k})k + \omega(\bar{k}) - k'\right) + \beta\delta V(k', \bar{k}') \right\}$

attained at $g(k,\bar{k})$

(2) Given $g(k,\bar{k})$, $V(k,\bar{k}) = \log\left(r(\bar{k})k + \omega(\bar{k}) - g(k,\bar{k})\right) + \delta V\left(g(k,\bar{k}), \bar{k}'\right)$

(3) $f'(\bar{k}) = r(\bar{k})$

$\omega(\bar{k}) = (1-\alpha) f(\bar{k})$ > factor prices determined competitively

(4) $g(\bar{k},\bar{k}) = G(\bar{k})$ (from consumers being identical)

Planner's Problem

$$V_0(k) \equiv \max_{k'} u\left(f(k) - k'\right) + \beta\delta V(k')$$

$$V(k) = u\left(f(k) - h(k)\right) + \delta V(h(k))$$

$\Rightarrow$ FONC $\quad u'\left(f(k) - h(k)\right) = \beta\delta V'(h(k))$

$$u(c)=\log(c)$$
$$f(k)=Ak^{\alpha}$$

$$V'(k)=u'(f(k)-h(k))(f'(k)-h'(k))+\delta h'(k)V'(h(k))$$

$$u'(f(k)-h(k))=\beta\delta\left\{u'(f(h(k))-h(h(k))\left[f'(h(k))-h'(h(k))+\frac{1}{\beta}h'(h(k))\right]\right\}$$
$$=\beta\delta\left\{u'(\underbrace{f(h(k))-h(h(k))}_{(1-m)f(h(k))})\left\{f'(h(k))+\left(\frac{1}{\beta}-1\right)h'(h(k))\right\}\right\}$$

Posit: $h(k)=mf(k)$
$$=mAk^{\alpha} \qquad h'=\alpha mAk^{\alpha-1}$$

$$\frac{1}{(1-m)Ak^{\alpha}}=\beta\delta\frac{1}{(1-m)A(mAk^{\alpha})^{\alpha}}\left\{\alpha A(mAk^{\alpha})^{\alpha-1}+\left(\frac{1}{\beta}-1\right)\alpha Am(mAk^{\alpha})^{\alpha-1}\right\}$$
$$=\beta\delta\frac{1}{(1-m)\cancel{A}(mAk^{\alpha})}\left(\alpha\cancel{A}+\left(\frac{1}{\beta}-1\right)\alpha\cancel{A}m\right)$$

$$1=\cancel{\beta}\delta\frac{1}{m}\left(\frac{\alpha}{\cancel{\beta}}\right)?$$
$$m=\alpha\delta? \qquad m=\frac{\alpha\beta\delta}{1-\delta(1-\beta)\alpha}$$

$$h(k)=\frac{\beta\delta}{1-\alpha\delta(1-\beta)}\alpha Ak^{\alpha}$$

$\cancel{u'((1-m)z)=\beta\delta V'(mz)}$ $\qquad z=mAk^{\alpha}$

$h(k)=z \qquad u'\left(\frac{z}{m}-z\right)=\beta\delta V'(z)$

$$V'(z)=\frac{1}{\beta\delta}\frac{1}{z(\frac{1}{m}-1)} \quad \Rightarrow V(k)=\log(z)\left(\frac{1}{\beta\delta(\frac{1}{m}-1)}\right)+C$$

Planner gives rise to lower savings.

$$g(k,\bar{k}) = \frac{\beta\delta}{1-\delta(1-\beta)} r(\bar{k})k$$

$$= \frac{\beta\delta}{1-\delta(1-\beta)} \alpha A k^{\alpha} \qquad \bar{k}=k$$

$$\frac{\beta\delta}{1-\delta(1-\beta)} > \frac{\beta\delta}{1-\delta(1-\beta)\alpha} \quad \text{since } \alpha\in(0,1).$$

Planner playing against future planners, none is a price taker.
Consumer playing against future versions of himself, all are price takers.

Planner Euler eq.: $u'(c_t) = \beta\delta u'(c_{t+1})\left(f'(k_{t+1}) + \underbrace{\left(\frac{1}{\beta}-1\right)h'(k_{t+1})}\right)$

$\beta<1 \Rightarrow$ positive term

$\beta=1$ back to normal Euler eq.

$\Rightarrow$ need to adjust c_{t+1} accordingly.

$u'(c_t)\uparrow \Rightarrow$ more savings than $\beta=1$.

Competitive eq. Euler:

$$u'(c_t) = \beta\delta u'(c_{t+1})\left(f'(k_{t+1}) + \left(\frac{1}{\beta}-1\right)g_1(k_{t+1},k_{t+1})\right)$$

$h'(\cdot)$ decreasing in k
g_1 linear in k $\Rightarrow$ more savings in competitive equilibrium
g_1 = constant

(1) Heterogeneity? Importance?
(2) Does uncertainty make a real difference?

Two countries fishing from lake.
$y_{t+1} = f(y_t)$ production function of fish (reproduction)

People $h = 1, 2, \dots, H$
Consumption / savings over time
$u_h(c_h)$
$0 < \delta_h < 1$ discounting
labor exogenously supplied
$y_t = w_t + (1+r_t)x_{t-1}^h$ w_t = wage at t
x_{t-1}^h = savings from last period

Perfect Foresight: Given $\{(w_t, r_t)\}_{t=1}^{\infty}$, $V^h(x) = \max \left\{ \sum_{t=1}^{\infty} \delta_h^{t-1} u^h(c_t^h) \;\middle|\; \begin{array}{l} x_0 = x \\ x_{t+1}^h = w_t + (1+r_t)x_t^h - c_t^h \\ x_t^h \geq 0, c_t^h \geq 0 \end{array} \right\}$

no borrowing against future income

Assume $1 > \delta_1 > \delta_2 > \delta_3 > \dots > \delta_H > 0$.
$\{(\bar{x}_t^h, \bar{c}_t^h)\}$ optimal path.

$$\max_{x_t^h} u^h\left(w_t + (1+r_t)\bar{x}_{t-1}^h - x_t^h\right) + \delta_h u^h\left(w_{t+1} + (1+r_{t+1})x_t^h - \bar{x}_{t+1}^h\right)\}$$

$$u^{h\prime}(\bar{c}_t^h) \geq \delta_h (1+r_{t+1}) u^{h\prime}(\bar{c}_{t+1}^h) \quad \forall t \quad \text{Euler eq.}$$
$$= \text{ if } \bar{x}_t^h > 0$$

CRTS production function $F(K,L)$.

$f(k)$ per capita production function

$f'>0,\ f''<0$

$$x_t = \sum_{h=1}^{H} x_{t-1}^h$$

$$1+r_t = f'(x_t) \qquad \omega_t = \frac{1}{H}\left(f(x_t) - f'(x_t)\,x_t\right)$$

Adding heterogeneity...

$\{(\bar{x}_0^h), (\bar{x}_t^h, \bar{c}_t^h), (\omega_t, r_t)\}$ equilibrium (Ramsey)

if (1) Consumers optimize given prices, wages, interest rates

(2) Firms maximize profits

(3) Markets clear.

Steady-state here is $\{(x^h, c^h), (\omega, r)\}$ if $x = \sum_h x^h$

$1+r = f'(x)$

$\omega = \frac{1}{H}\left[f(x) - f'(x)x\right]$

$x^1 = x$

$x^h = 0 \quad h \geq 2$ in steady state

most patient person gets all the capital.

$\delta_1(1+r) = 1$

$\delta_h(1+r) < 1$ for $h \geq 2 \Rightarrow x^h = 0$

People $2, \ldots, H$ consume only their wages.

Only person 1 saves.

- Heterogeneous agents with discount factors $1 > \delta_1 > \delta_2 > \cdots > \delta_H > 0$.

$K_1 < K_2$, $u_1 = u_2 = u_3 = \cdots$ $\quad \{K_2, K_1, K_2, K_1, \ldots\}$

$\delta_1 f'(K_1) u'(c_2^1) = u'(c_1^1)$

$\delta_1 f'(K_2) u'(c_1^1) = u'(c_2^1)$ $\quad$ Person 1 $\rightarrow (c_2^1, c_1^1)$

For rest of economy, $\delta_H f'(K_1) u'(\omega_2) \leq u'(\omega_1)$

$\delta_H f'(K_2) u'(\omega_1) \leq u'(\omega_2)$ $\quad \forall h \neq 1.$ $\quad c_t^h = \omega_t + (1+r_t) x_{t-1}^h$

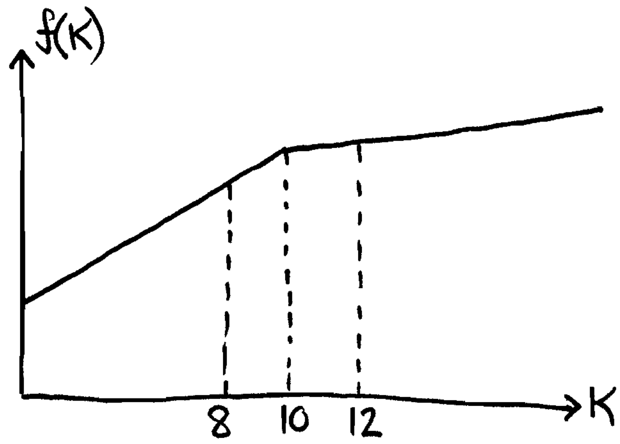
Capital stock becomes owned by person 1.

$\cancel{c^1 = f(}$

Example:

$H = 2$

$\delta_1 > \delta_2$



$f(K) = \begin{cases} 10 + 5K, & K \leq 10 \\ 52 + 0.8K, & K \geq 10 \end{cases}$

Let $K_1 = 8, K_2 = 12$

$K_1 f'(K_1) = 8 \cdot 5 = 40$

$K_2 f'(K_2) = 12 \cdot 0.8 = 9.6$

$f(K_2) - K_2 f'(K_2) = 61.6 - 9.6 = 52 = 2\omega_1$

$\Rightarrow \omega_1 = 26$

$f(K_1) - K_1 f'(K_1) = 50 - 40 = 10 = 2\omega_2$

$\Rightarrow \omega_2 = 5$

$c_1^1 = \bar{f}(K_2) - K_1 = 27.6$

$c_2^1 = \bar{f}(K_1) - K_2 = 33$

$$c^1 = \underbrace{\frac{1}{H}\left[f(K) - K f'(K)\right] + K f'(K)}_{\bar{f}(K)} - K'$$

Condition on utility: $u'(27.6) = 1$

$u'(33) = 0.4$

With two different discount factors, we can get cycles in a one-sector economy.

Steady-state at $K=10$

When would we get convergence of Ramsey eq. to unique steady state? $\frac{d}{dK}\left[K f'(K)\right] > 0$

$$\Leftrightarrow \quad -\frac{K f''(K)}{f'(K)} < 1$$

Look at handout.

3 Certainty vs. Uncertainty: An Example.

As was shown for the certainty case by Becker (1980), in a steady state Ramsey equilibrium we must have:

$$X^1 = \overline{K} \textit{ and } X^h = 0 \textit{ for } h = 2, ..., H$$

where $\overline{K}$ solves:

$\max_K [f(K) - K(1 + \overline{r})]$ where $1 + \overline{r} = 1/\delta_1$.

Let us demonstrate now that this result breaks down in the *stochastic* model.

Consider the following example with $H = 2$. The stochastic production function is given by:

$$f(X, \tilde{\theta}) = \tilde{\theta} X^\alpha, \quad \text{where } 0 < \alpha < 1.$$

The utility functions of individuals 1 and 2 are:

$$u_1(c) = \ln c$$

$$u_2(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0.$$

$\omega = (\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots)$ realization of $\tilde{\Theta}$

$X_t(\omega)$ function depends on past and current realizations of ω.

If $X_t(\omega) = X(\omega)$, then it is stationary. $X_t(\omega) = X(T^t \omega)$ for all ω.

$(T\omega)_K = \omega_{K+1}$, $K = \ldots, -1, 0, 1, \ldots$ T shift operator

Also let $1 > \delta_1 > \delta_2 > \frac{1}{2}$, and $\alpha < 1/(2\delta_1 - 1)$.

Assume now that the Becker (1980) result for deterministic stationary Ramsey equilibrium holds in the stochastic case as well. Let $< X_1^*(\omega),\ X_2^*(\omega),\ c_1^*(\omega),\ c_2^*(\omega);\ W(\omega),\ r(\omega) >$ be a steady state in this economy. By our assumption since $\delta_1 > \delta_2$ we have $X_2^*(\omega) = 0$ a.s., therefore we obtain the following expressions for W and r:

almost surely

$$W(T\omega) = \frac{1}{2}\left[f(X_1^*(\omega), \omega_1) - X_1^*(\omega) f\prime(X_1^*(\omega), \omega_1)\right] = \frac{1-\alpha}{2}\omega_1 \left[X_1^*(\omega)\right]^\alpha \text{ a.s.} \quad (9)$$

$$1 + r(Tw) = \alpha\omega_1 \left[X_1^*(\omega)\right]^{\alpha-1} \text{ a.s.} \quad (10)$$

The budget equation for individual 1 can be written in this example as:

$$X_1^*(\omega) + c_1^*(\omega) = \frac{1+\alpha}{2}\omega_0 \left[X_1^*(T^{-1}\omega)\right]^\alpha \text{ a.s.} \quad (11)$$

Following the examples in class when the utility function is logarithmic and the production function is Cobb-Douglas, since $X(\omega) = X_1^*(\omega)$ a.s., the optimal consumption policy function of individual 1, $g_1(y)$, is linear in his beginning of period income y. Let us write, $g_1(y) = \lambda y$ and observe the following functional equation must hold for all y (note that this equation is, basically derived from equation (7) and $X_1^*(\omega) > 0$ a.s. in this case),

$$\frac{1}{\lambda\frac{1+\alpha}{2}\omega_0\left[X_1^*(T^{-1}\omega)\right]^\alpha} =$$

$$\delta_1 E_0\left[\alpha\omega_1\left((1-\lambda)\tfrac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^\alpha\right)^{\alpha-1} \frac{1}{\lambda\frac{1+\alpha}{2}\omega_1\left[(1-\lambda)\omega_0\frac{1+\alpha}{2}X_1^*(T^{-1}\omega)^\alpha\right]^\alpha}\right].$$

$$u_1'(c_t^*) = \delta_1 E_t\left[f'(X_t^*, \omega) u_1'(c_{t+1}^*)\right]$$

Hence,

$$\omega_0^{-1} X_1^*(T^{-1}\omega)^{\overset{-\alpha}{-1}} = \delta_1 \alpha E_0 \left[(1-\lambda)\omega_0 \frac{1+\alpha}{2} X_1^*(T^{-1}\omega)^{\alpha}\right]^{\overset{\alpha}{-\gamma}} \quad \text{a.s.}$$

which implies that:

$$\lambda = 1 - \frac{2\delta_1 \alpha}{1+\alpha}.$$

Since $1 > \delta_1 > \frac{1}{2}$ and $0 < \alpha < [2\delta_1 - 1]^{-1}$ we find that $0 < \lambda < 1$, hence $g_1(y) = \lambda y$ is a consumption policy function.

Consider now the optimization process of individual 2. Since, by our assumption, $X_2^*(\omega) = 0$ a.s. we have

$$c_2^*(\omega) = W(\omega) \text{ a.s.}$$

Namely,

$$c_2^*(\omega) = \frac{1-\alpha}{2}\omega_0 \left[X_1^*(T^{-1}\omega)\right]^{\alpha} \quad \text{a.s.}$$

Using the Euler conditions (8), when $X_2^*(\omega) = 0$ a.s. i.e., the inequality case,

$$u_2'(c_2^*(\omega)) \geqq \delta_2 E_0 \left[(1 + r(T\omega))\, u_2' \left(c_2^*(T\omega)\right)\right] \quad \text{a.s.}$$

We find for our case that

$$\left\{\frac{1-\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^{\alpha}\right\}^{-\gamma} \geqq \delta_2 E_0 \left\{ \begin{array}{c} \alpha\omega_1 \left[(1-\lambda)\frac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^{\alpha}\right]^{\alpha-1} \\ \cdot \left[\frac{1-\alpha}{2}\omega_1 \left((1-\lambda)\frac{1+\alpha}{2}\omega_0 X_1^*(T^{-1}\omega)^{\alpha}\right)^{\alpha}\right]^{-\gamma} \end{array} \right\}$$

Simplifying this inequality we reach:

$$\left[X_1^*(T^{-1}\omega)\right]^{-\alpha\gamma+\alpha-\alpha^2(1-\gamma)} \geqq \delta_2 \delta_1^{\alpha-1-\alpha\gamma} [\alpha]^{\alpha-\alpha\gamma} [\omega_0]^{\alpha+\gamma-1-\alpha\gamma} \cdot E\left[\omega_1^{1-\gamma}\right] \quad \text{a.s.}$$

Let us assume now that $\omega_t \in [a,b]$,0<a<b<∞. By choosing $\gamma > 0$ but small and $\alpha < \frac{1}{2}$ we can guarantee that

(i) $\Lambda = -\alpha\gamma + \alpha - \alpha^2(1-\gamma) \geqq \alpha/2$.

(ii) $\alpha + \gamma - 1 - \alpha\gamma < 0$.

(For example: $\gamma = 1/16 \quad \alpha = 1/4$). Thus,

$$\left[X_1^*(T^{-1}\omega)\right]^{\Lambda} \geqq \delta_2\delta_1^{\alpha-1-\alpha\gamma}[\alpha]^{\alpha-\alpha\gamma}E_0\left[\omega_1^{1-\gamma}\right][a]^{\alpha+\gamma-1-\alpha\gamma} \text{ a.s.} \tag{12}$$

However, it is known in the stochastic growth literature that for any given $\in > 0$, by choosing "a" small we can assure that

$$\text{Prob}\left\{X_1^* < \in\right\} > 0.$$

Thus condition (12) cannot hold with probability 1. This contradiction demonstrates that our assumption that in this equilibrium $X_2^* = 0$ a.s. cannot be true. Hence it proves our claim.

The example shows that in contrast to the stark distribution of steady state capital obtained in the deterministic model, the introduction of uncertainty can lead to positive saving for the relatively impatient agent 2 for a set of states with positive measure. The presence of technological shocks means that the impatient agent has a buffer stock motive for holding capital that is absent from the deterministic story. Thus, we cannot conclude that one consumer will *always* have *all* the economy's capital. This implies that the stationary state is not determined by a single agent's dynamic optimization problem as in the deterministic model. Therefore the establishment of a stochastic stationary equilibrium for this economy must take into account the interactions of all the agents optimization problems (including the profit conditions). This naturally leads us to a fixed point argument for our existence result.