①

Audit - if no very involved

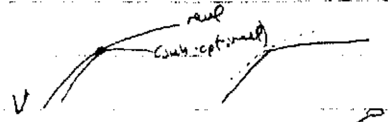
P.Sets - groups - publication record.

Econ 609 (Lec)

09/01/2008.

Value f^n - proving differentiability

- std. way is to get concavity & then rule out kinks by bounding the value f^n below by a smooth f^n.



floor on how good a good job will be

Value f^n vs. sloppy

V - at least as good as doing it sloppy

it equal here and bounded below ↳ cannot be kinked

Bellman Equation:

unmaximized value one stage

stochastic gross rate of return

$$V_t(W) = \max\ u(c_t) + \beta E_t\left[V_{t+1}\left(\tilde{R}_{t+1}(W_t - c) + \tilde{Y}_{t+1}\right)\right]$$

starting wealth $c \in [0, W_t]$

↳ non-capital, exogenous income (stochastic)

W_{t+1}

$= \max\ F(W_t, c)$ - f^n of state & control variable (C)

How to prove properties of V? (Backwards Recursion)

① $V^{T+h} \in P$ (V has some property P — e.g. nondecreasing ↗) $V_{T+1}(W_{T+1}) \equiv 0$

② Recursion

a) Show $V^{t+h} \in P \Rightarrow F^t(K_t, X_t) \in Q$

f^n of time var — state vbls (vector) — f^n of var. vars — control vbls (vector)

T+1 — T — T-1 — T-2

If true at T+1, true at T

(Usually P and Q related)

b) show $F^t \in Q \Rightarrow V^t \in P$ (preservation under maximization)

For our case

$$V^t(K_t) = \max_{X_t} F^t(K_t, X_t) \quad ; \quad V_t^t(W_t) = \max_c F^t(W_t, C)$$

$V^t(\cdot)$

Need to behave ok under maximizations and expectations to extend to property in $F(\cdot,\cdot)$

Version of Subgame Perfection ↓ need to trade off today and the best you can get (V_{t+1}) from what's leftover.

Lifetime utility f^n: $V_0 = E_0 \sum_{t=0}^{T} \beta^t u(c_t)$

the thing the agent is maximizing (obj. f^n) $\Leftrightarrow V_t = u(c_t) + \beta E_t V_{t+1}$

Proof: $V_0 = E_0(u(c_0) + \beta u(c_1) + \cdots)$

$= E_0(u(c_0) + \beta[u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \cdots])$

$E(x) = E(E(x|z))$ LIE $= u(c_0) + E_0\beta[u(c_1) + \beta E_1 \cdots]$.

②

$K_{t+1} = \Gamma(K_t, X_t, \text{Random Vbls})$ - evolution of state variable

In our example: $W_{t+1} = \tilde{R}_{t+1}(W_t - c) + \tilde{y}_{t+1}$

Prove V increasing, concave

$V_W \geq 0$ - more money better than less

$V_{WW} \leq 0$

$V_{Wp} \geq 0$ (if introduce $V_{t+1}(\ldots \tilde{y}_{t+1}; \beta))$)

treating parameter as argument

structure of the problem

Base case: V_T - always that

Strictly Increasing

Claim

$V_t(W+\delta) - V_t(W) \geq 0 \quad \forall \delta > 0$

no need to subscript

Proof: $V^t_{\$}(W_t+\delta) - V^t_{\$}(W_t)$ (W is a dummy var.)

$$= \max_{c\in[0,W_t+\delta]} U^t(c) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W_t+\delta-c) + \tilde{y}_{t+1})$$

different c's!

$$- \max_{c\in[0,W_t]} U^t(c) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W_t-c) + \tilde{y}_{t+1})$$

let c^* be the argmax here

could be bigger than 1

sloppy choice - need c^* to be a feasible choice - which it is

larger input value - since V^{t+1} ↑ - this diff ≥ 0.

$$\geq u^t(c^*) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W_t+\delta-C^*) + \tilde{y}_{t+1})$$

$$- u(c^*) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W_t - C^*) + \tilde{y}_{t+1}) \overset{?}{\geq} 0$$

preserved under expectation as well

We know it's true for t+1 b/c we're doing recursion!

$\to V^{t+1}(W_{t+1}+\delta) - V^{t+1}(W_{t+1}) \geq 0$ (Recursion Hypothesis)

$$= \beta\left(E_t\left\{V^{t+1}\left[(\tilde{R}_{t+1}(W_t+\delta-c^*) + \tilde{y}_{t+1}\right]\right\} - E_t V^{t+1}\left[(\tilde{R}_{t+1}(W_t-c^*) + \tilde{y}_{t+1})\right]\right)$$

To get strictness - we need perversion (T)

Goes through if you trust yourself your future self to maximize

Note: no property of utility f^n that matters

-> could you prove in the same way that value f^n decreasing?

If choose $\bar{c}$ to max by one:

$$\leq u^t(\bar{c}) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W+\delta-\bar{c}) + \tilde{y}_{t+1})$$

$$- u(\bar{c}) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(W-\bar{c}) + \tilde{y}_{t+1})$$

b/c $\bar{c} \notin [0, W_t]$

But now $\bar{c}$ is not in the interval!

See the issue w/ $V_t(W-\delta) - V_t(W)$ - same issue - not in the set

∴ Showed

$$V^t(W+\delta; T) - V^t(W; T) \geq 0 \quad \text{(worked for all finite } T\text{)}$$

Suppose

$$\lim_{T\to\infty} V^t(W, T) = V^t(W, \infty) \quad \text{(limit exists)}$$

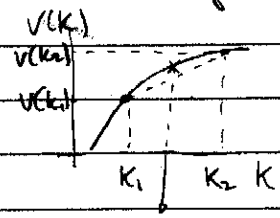
then $V^t(W+\delta, \infty) - V^t(W, \infty) = \lim_{T\to\infty}\left[V^t(W+\delta, T) - V^t(W, T)\right] \geq 0$

(Limit of the diffs is the diff. of lim) ~~is limit of diff~~

Note: did not need any assumption on utility f^n.

$$V^t(k_t) = \max_{x} F^t(k_t, x)$$
$$k_{t+1} = \Gamma^t(k_t, X, \omega_{t+1})$$

To prove concavity:



- Above secant line - nice discrete way of thinking about it

$\theta k_1 + (1-\theta) k_2$ w/ $\theta \in [0,1]$

Claim: $(A) = V^t(\theta k_1 + (1-\theta)k_2) - \underline{\theta V(k_1) - (1-\theta) V(k_2)} \geq 0$.

secant line - weighted average of the f^n at endpoints at endpts

$$\&\ (A) = \max_{x \in X(\theta k_1 + (1-\theta)k_2)} F^t(\theta k_1 + (1-\theta)k_2, x) - \theta \max_{x\in X(k_1)} F^t(k_1, x) - (1-\theta)\max_{x\in X(k_2)} F^t(k_2, x)$$

$$\geq F^t(\theta k_1 + (1-\theta)k_2, \hat{x}) - \theta F^t(k_1, x_1) - (1-\theta) F^t(k_2, x_2)$$

w/ $\hat{x} \in X(\theta k_1 + (1-\theta)k_2)$

w/ $x_1 \in \arg\max_{x \in X(k_1)} F^t(k_1, x)$ and $x_2 \in \arg\max$

negative sign caused to do it right; if ⊕ sign - don't apply

What $\hat{x}$, and property of F do we need to show that this ≥ 0?

$\frac{f'(Q)Q}{f(Q)}$ $\quad$ (1) $\quad \frac{dP}{dQ}\frac{Q}{P}$ $\quad \frac{\partial f(Q)}{\partial Q}\frac{Q}{P}$

$$\Pi = f(Q)Q - C(Q)$$

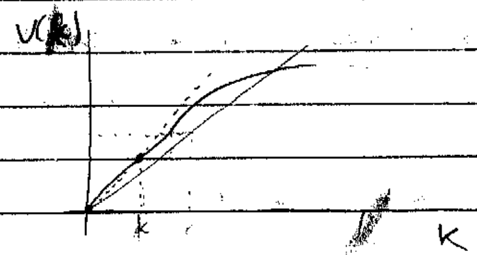
$$\frac{\partial \Pi}{\partial Q} = f'(Q)Q - c'(Q) + f(Q) = 0$$

Econ 609 (Lec) $\quad$ 14/01/2007

Course Website: Kimball - risk

1st Homework: (1) In a firm's problem find and prove reasonable & fairly general conditions under which the value f^n has decreasing returns to scale

Present - Max



Hugo - CRS (symmetry)

Due in one week

If I double K, V(k) ↑ by less than 2

DRTS: $V(\theta k) - \theta V(k) \geq 0 \quad \forall\, \theta \in [0,1]$ $\qquad$ CRTS $V(\theta k) - \theta V(k) = 0 \quad \forall \theta \geq 0$

IRTS: $V(\theta k) - \theta V(k) \geq 0 \quad \forall\, \theta \geq 1$

DRTS: $\frac{\partial \ln V(k)}{\partial \ln k} < 1$

$\frac{V'(k)}{V(k)\left(\frac{1}{k}\right)} \Rightarrow \frac{kV'(k)}{V(k)} \leq 1$ $\quad$ (First derivative property, not 2nd derivative - so not concavity!)

(2) IRTS

Back to last lecture:

$$V(\theta k_1 + (1-\theta)k_2) - \theta V(k_1) - (1-\theta)V(k_2)$$

$$= \max_{\substack{x \text{ s.t.} \\ (\theta k_1 + (1-\theta)k_2,\, x) \in \mathcal{W}}} F(\theta k_1 + (1-\theta)k_2, x) - \theta \max_{\substack{x \text{ s.t.} \\ (k_1, x)\in\mathcal{W}}} F(k_1, x) - (1-\theta)\max_{\substack{x \text{ s.t.} \\ (k_2,x)\in\mathcal{W}}} F(k_2, x)$$

$\mathcal{W}$ - set of feasible things

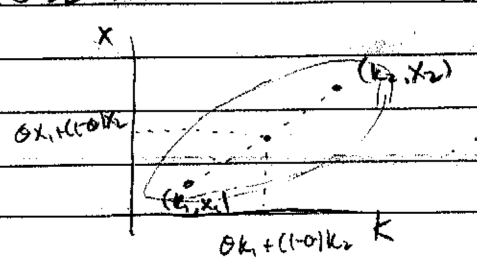
$$\geq F(\theta k_1 + (1-\theta)k_2, \hat{x}) - \theta F(k_1, x_1) - (1-\theta)F(k_2, x_2)$$

w/ $x_1 \in \operatorname{argmax}_{\substack{x \text{ s.t.} \\ (k_1,x)\in\mathcal{W}}} F(k_1, x_1)$

let $\hat{x} = \theta x_1 + (1-\theta)x_2$

Now, we need joint concavity of $F(\cdot,\cdot)$ which we will show later

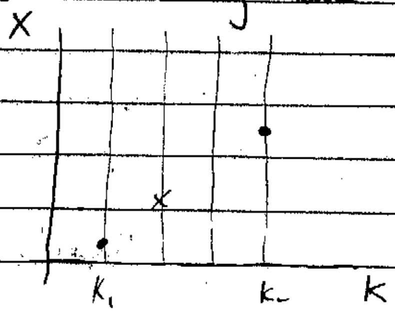
is $\hat{x} \in \mathcal{W}$?:



need $\mathcal{W}$ convex - but that's stronger than you need.

PTO →

Suppose K can only take on integer values, $X \in \mathbb{R}$



not convex in $\mathbb{R}^2$

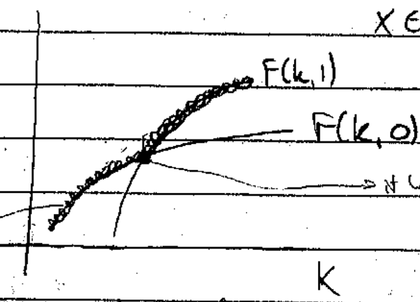
Still works whenever $\theta K_1 + (1-\theta)K_2$ is a feasible choice of K

∴ Just need the value at X to be in the set.

Important Problem

What if X discrete?

$X \in \{0, 1\}$



• Soldier of Fortune

→ if here – gamble to get higher utility

Lower-concavity can make ppl risk loving!

Value f^n – NOT concave (even though $F(.)$ concave)

∴ Discrete choices can lead to non-concave reward (value f^n's)

• Proving joint concavity of $F(.,.)$ in K, X:

Eg: $$V^t(W) = \max_c u(c) + \beta E_t V_{t+1}(\tilde{R}_{t+1}(W-c) + q_{t+1})$$

V^{t+1} – concave; taking concave of linear – concave (recursion hypothesis)

$V^t \equiv 0 \ \forall t > T$ E_t – adding up concave f^n's – concave

need $u(c)$ concave in c.

[Writing it recursively assumes no time inconsistency]

Firm's Problem

(*) $$V^{t}(K_t, Z_t) = \max_{X} \underbrace{\Pi(K)}_{\text{profit}} - \underbrace{P_I C(X, K, Z_t)}_{\text{pay for investment}} + E_t[D^{t+1}(Z_{t+1}) V^{t+1}(X, Z_{t+1})]$$

Z_t: exogenous state vars (omitting parameters)

$X = K_{t+1}$ (state variable in future)

$D^{t+1}(Z_{t+1})$ → stochastic discount factor

not much action here ∴ $X \in \mathbb{R}^+$

Choosing the constraint set is not a big issue here.

What properties on $\Pi(.)$, $c(.)$

What if $\max_{K,H \in \Theta^+} \Pi(K,H)$ *

Max over H first

$$\hat{\Pi}(K) = \max_H \Pi(K,H)$$, and stick it in for $\Pi(K,H)$ to make the problem identical to (*)

Optimization sub-problem: need the right amt. of H each period

What kind of factor mkt / prod^n f^n will deliver ~~this kind~~ the kind of $\Pi(\cdot)$ you require (prod^n f^n, factor mkt, product mkt structure)

(3) Think about what economic environment you need, to get $\Pi(K)$ of the slope you need

downward sloping demand will affect $\Pi(\cdot)$

$$V^t(w, \beta) = \max_c u(c) + \beta E_t V^{t+1}(\underbrace{\tilde{R}_{t+1}(w-c) + q_{t+1}}_{w_{t+1}}; \beta)$$

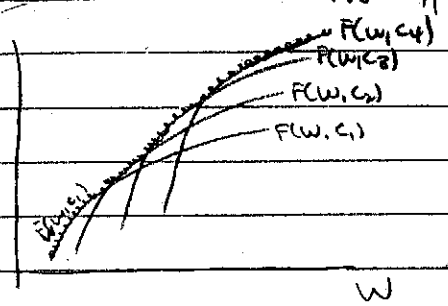
$\uparrow \beta$; consume less

FOC: ∂c: $u'(c) - \beta E_t \tilde{R}_{t+1} V^{t+1}_w(\tilde{R}_{t+1}(w-c) + q_{t+1}; \beta) = 0$

$V^t_w(w,\beta) = \beta E_t \tilde{R}_{t+1} V^{t+1}_w(w_{t+1};\beta)$ $\therefore u'(c) - \beta E_t \tilde{R}_{t+1} V^{t+1}_w(\tilde{R}_{t+1}(w-c) + q_{t+1}; \beta)$

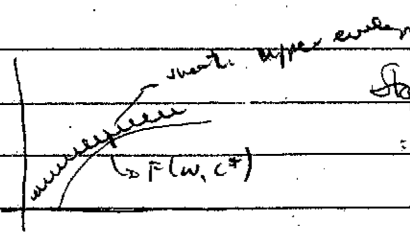
$$u'(c) = V^t_w(w, \beta) \quad \text{(Envelope Theorem)}.$$

E.T:



Slope of upper envelope = slope of constituent curve

For continuous choice of c:



~~Slope at max~~ = ~~slope~~

What property of value f^n will ~~make~~ mean $\beta \to c$?

$\rightarrow$ $V_{w\beta} \geq 0$ (Supermodularity of V in w and β)

$\uparrow \beta$ if $V^t_w(w,\beta) \uparrow \Rightarrow c \downarrow$ $u'(c) \uparrow$

Def^n. (of Supermodularity)

K is a big vector

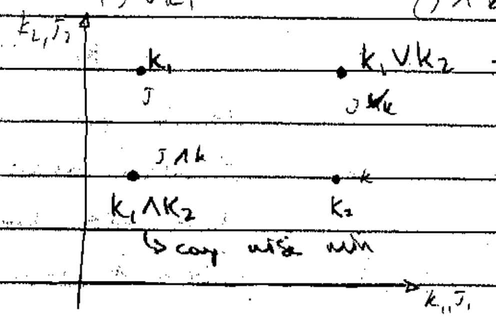
PTO →

"join" "meet" → discrete version of cross partial

$$\overset{\text{"join"}}{V(k_1 \vee k_2)} + \overset{\text{"meet"}}{V(k_1 \wedge k_2)} - V(k_1) - V(k_2) \geq 0$$

$(J \vee K)$ $(J \wedge K)$ J K

$\mathbb{R}^2$



$\vee \approx \cup$ - big (union)

$\wedge \approx \cap$ - intersection - small

Joint supermodularity for n components = joint supermodularity for all pairs of components

$K_1 = (H_1, J_1)$ Vectors of components

$K_2 = (H_2, J_2)$

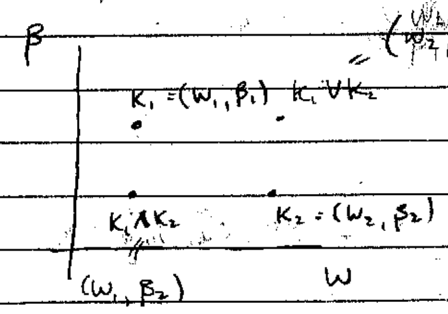
ECON 609 (lec) 16/01/2008

- Outline lock for proving supermodularity apart derivative (differentiability)

let's show $F(\cdot,\cdot)$ supermodular, and then $V(\cdot)$ supermodular.

Define : $S = W - C$ — endogenous part of w next period.

$K = \begin{pmatrix} w \\ \beta \end{pmatrix} : \begin{pmatrix} K \\ X \end{pmatrix} = \begin{pmatrix} w \\ \beta \\ s \end{pmatrix}$ → component-wise max



⊕ interaction b/w wealth and β

Components have meaningful names.

→ ○ if you can rank k_1, k_2 similarly $k_1 \vee k_2$ across all components

Show $V(\cdot)$ supermodular

Claim: $V(K_1 \vee K_2) + V(K_1 \wedge K_2) - V(K_1) - V(K_2) \geq 0$

Proof: $V(K_1 \vee K_2) + V(K_1 \wedge K_2) - V(K_1) - V(K_2)$

lets it be → it 7 sum

$$= \max_{(K_1 \vee K_2, X) \in \mathcal{W}} F(K_1 \vee K_2, X) + \max_{(K_1 \wedge K_2, X) \in \mathcal{W}} F(K_1 \wedge K_2, X) - \max_{(K_1, X) \in \mathcal{W}} F(K_1, X) - \max_{(K_2, X) \in \mathcal{W}} F(K_2, X)$$

$$\geq \max_{(K_1 \vee K_2, X) \in \mathcal{W}} F(K_1 \vee K_2, X) + \max_{(K_1 \wedge K_2, X) \in \mathcal{W}} F(K_1 \wedge K_2, X) - F(K_1, X_1) - F(K_2, X_2)$$

w/ $X_1 \in \arg\max_{(K_1, X) \in \mathcal{W}} F(K_1, X)$

3 things: • sloppy choice of X
- properties of $F(\cdot, \cdot)$ — Supermodularity in all pairs is supermodularity in the large dimension
- set $\mathcal{W}$

- Assume F s.m. in $\begin{pmatrix} k \\ x \end{pmatrix}$

• sloppy choice of X

$$\geq F(K_1 \vee K_2, X_1 \vee X_2) + F(K_1 \wedge K_2, X_1 \wedge X_2) - F(K_1, X_1) - F(K_2, X_2)$$

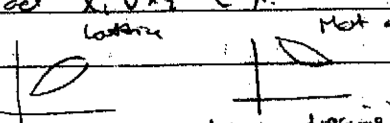
$F((K_1, X_1) \vee (K_2, X_2))$ $F((K_1, X_1) \wedge (K_2, X_2))$, $F(z_1 \wedge z_2)$, z_1, z_2

We need $F(\cdot,\cdot)$ supermodular in big vector!

≥ 0 by defn of supermodularity

Lattice: if $x_1, x_2 \in \mathcal{X}$ then $x_1 \wedge x_2$ and $x_1 \vee x_2 \in \mathcal{X}$.

- This is what we need $\mathcal{W}$



(Generates and needs diagrams) PTO

② why subscripted? Supermodularity of F in w, β, c, is diff than in w, β, s.

good to relate to firm's state

Freedom in defining controls

$(S = W - C)$

Proving that $F(\cdot,\cdot)$ is supermodular - need to show all cross derivatives are ≥ 0

$$V^{t}(w,\beta) = \max_{S} \underbrace{u(w-s) + \beta E_t V^{t+1}(\tilde{R}_{t+1}(s) + \tilde{y}_{t+1}, \beta)}_{F(w,\beta,S)}$$

$$F_w = u'(w-s) + \cancel{\beta E_t V_w^{t+1}(w')}$$

$$F_\beta = E_t V^{t+1}(\tilde{R}_{t+1} S + \tilde{Y}_{t+1}, \beta) + \beta E_t V_\beta^{t+1}(\tilde{R}_{t+1} S + \tilde{Y}_{t+1}, \beta)$$

$$F_s = -u'(w-s) + \beta E_t \tilde{R}_{t+1} V_w^{t+1}(\tilde{R}_{t+1} S + \tilde{y}_{t+1}, \beta)$$

1st arg.

$$F_{w\beta} = 0$$

$$F_{ws} = -u''(w-s) \geq 0$$

gross rate of return (limited liability) ≥ 0

To prove

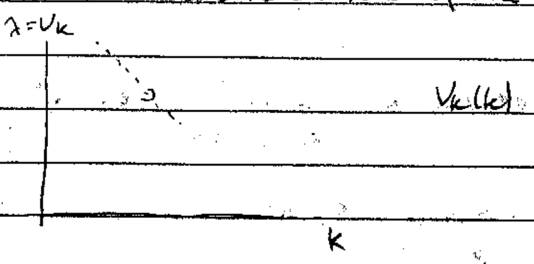
$$F_{\beta s} = F_{s\beta} = \underbrace{E_t \tilde{R}_{t+1} V_w^{t+1}(\tilde{R}_{t+1} S + \tilde{Y}_{t+1}, \beta)}_{\oplus \text{ from prev proof}} + \underbrace{\beta E_t \tilde{R}_{t+1} V_{w\beta}^{t+1}(\tilde{R}_{t+1} S + \tilde{Y}_{t+1}, \beta)}_{\oplus \text{ by recursive hypothesis}} \geq 0$$

$V_{w\beta} \geq 0$

Even though lots of stochastic processes going on, more patient will lower consumption

At T: $V^T(w,\beta) = \max u(w-s) \Rightarrow V^T_{w\beta} = 0$

Business Cycle Models (Social Planner's Problem of unique steady state)



if $V_{kk} \leq 0$ - downward sloping saddle path

$V_{k\beta} \geq 0$

If c instead of s, it would not have worked!

Back to the firm's problem

shifts Πf^n

Price of what you - X, with it adjustment cost f^n

$$V^{t+1}(k_{t+1}, \alpha, \beta, \gamma, z_t) = \max_{X} \Pi(k, \alpha, z) - P_z(\gamma, z_t) C(X, K, z_t) + \underbrace{E_t D^{t+1}(\beta, z_{t+1}) V^{t+1}(X, \alpha, \beta, \gamma, z_{t+1})}_{F}$$

$X = K_{t+1}$ ss

$z_{t+1} = \Gamma(z_t, \omega_{t+1})$

FOC: $-P(\gamma, z_t) C_x(X, K, z_t) + E_t D^{t+1}(\beta, z_{t+1}) V_k^{t+1}(X, \alpha, \beta, \gamma, z_{t+1}) = 0$

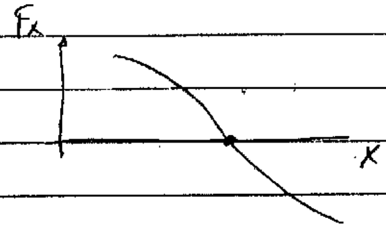
1st arg

$$\therefore P(\gamma, z_t) C_x(X, k, z_t) = E_t D^{t+1}(\beta, z_{t+1}) V_k^{t+1}(X, \alpha, \beta, \gamma, z_{t+1})$$

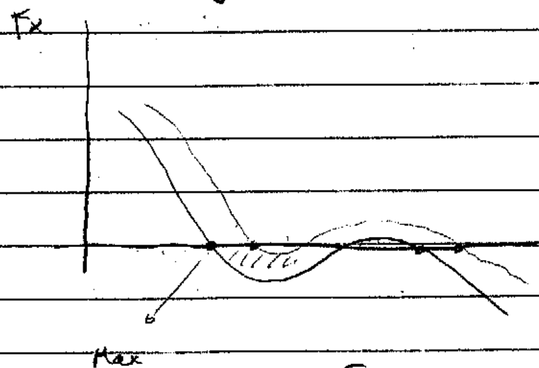
If $V_{xx} \geq 0$

w/ $C_{xx} \geq 0$

Monotone Comp. Statics:



No concavity. $\underline{\underline{\alpha}}\uparrow$



likely to jump - max will jump around.

Milgrom + Shannon

$F_{x\alpha} \geq 0$ (like supermodularity)

∘ Strong Set Order.

Don't need concavity to shift max to the right.

α like demand.

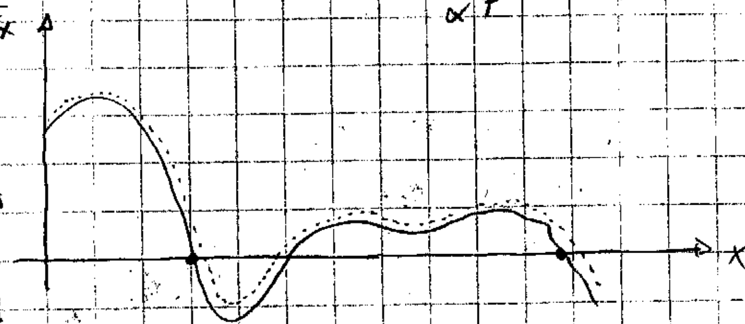
<u>Econ 609 (lec)</u> all other state variables ① 23/01/2008

$$V^t(k_t, \alpha, \beta, \gamma; z_t) = \max_x \{ \Pi(k, \alpha, z_t) - P(\gamma, z_t) C(x, k; z_t) + E_t D(\beta, z_{t+1}) \cdot V^{t+1}(x, \alpha, \beta, \gamma, z_{t+1}) \}$$

$z_{t+1} = \Gamma(z_t, w_{t+1})$ (z exogenous) $V(k,\alpha,\beta,\gamma) = \max_x F(k,\alpha,\beta,\gamma,x)$

FOC: $-P(\gamma, z_t) \cdot C_x(x, k; z_t) + E_t D(\beta, z_{t+1}) V_k^{t+1}(x, \alpha, \beta, \gamma; z_{t+1})$

$V_{k\alpha} \geq 0$ $\alpha \uparrow$



Suppose $F_{x\alpha} \geq 0$, then $\alpha \uparrow \Rightarrow \uparrow F_x (\Rightarrow x \uparrow ?)$

Must move optimal values to the right

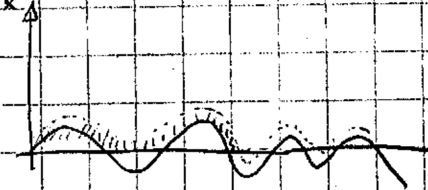
2nd order: $F_{xx} < 0$ - rules out some

Brute Force: Can evaluate at all candidate points

Candidate pt's move to the right, or vanish

If it disappears, clearly want to go farther to the right

- take next candidate point



Candidates shift to the right - don't need concavity

→ Concavity is needed for smooth moving of optimal value of x.

→ Here, x^* could jump around

Need to prove something about $V_{k\alpha} \geq 0$ to get $F_{x\alpha} \geq 0$. (which is what we want for forward (α) raises (x)) want at

Last period: $V^T(k_t, \alpha, z_t) = \max_x \Pi(k, \alpha, z_t) - P(\gamma, z_t) C(x, k, z_t)$ ② Could we not have got the $C_{kx} \leq 0$ condition from here?

① why don't we care here $V^{T+1} \equiv 0 \Rightarrow V_{k\alpha}^{T+1} = 0$ ✓

No reason to care about x b/c no period in T+1 - set x to lowest possible value

Might as well assume $\underline{\Pi_{k\alpha} \geq 0}$ (hopeless w/out this)

Need to worry about supermodularity k, α, x (larger set)

prove: If $\underset{\text{by induction}}{V_{k\alpha}^{t+1} \geq 0}$ then $F_{k\alpha} \geq 0$, $F_{x\alpha} \geq 0$, $F_{kx} \geq 0$ $\Rightarrow V_{k\alpha}^t \geq 0$

$\Rightarrow V_{k\alpha}^t \geq 0$

→ the two + the thing you're maximizing over.

(?) if $F_{k\alpha} \geq 0$, $F_{x\alpha} \geq 0$, $F_{kx} \geq 0$ then $V_{k\alpha}^t \geq 0$ · Preser Max Theorem (if supermodular jointly over all, then supermodular over subset of vars over max)

↳ Proof by citation - need supermodularity + lattice

↳ sufficient condition, not necessary

$F_{k\alpha} = \Pi_{k\alpha} \geq 0$

$F_{x\alpha} = E_t D(\beta, z_{t+1}) V_{k\alpha}^{t+1}(x, \alpha, \beta, \gamma, z_{t+1})$ (induction hypothesis)

$F_{kx} = -P(\gamma, z_t) C_{kx}(x, k, z_t) \geq 0$, need $C_{kx} \leq 0$

What does this mean?

if $\Pi_{k\alpha} \geq 0$ and $C_{kx} \leq 0$ then $V_{kx} \geq 0$ and $F_{x\alpha} \geq 0$.

$\Rightarrow \alpha \xrightarrow{+} x$.

No adjustment cost case: $x = (1-\delta)k + I$

$x = e^{-\delta h}k + hI$ (continuous time)

$\therefore I = x - (1-\delta)k$

Naturally $+p_0 C = -p_I I \Rightarrow C(k,x) = I = x-(1-\delta)k$

$\therefore C_{kx} = 0$

$kJ\left(\frac{I}{k}\right)$ $J'' \geq 0$ (Q-theory argument).

$C(x,k) = kJ\left(\frac{x - (1-\delta)k}{k}\right) = kJ\left(\frac{x}{k} - (1-\delta)\right)$

$C_x(x,k) = \cancel{k} J'\left(\frac{x}{k}-(1-\delta)\right) \cdot \frac{1}{\cancel{k}}$

$C_{kx} \geq 0$: $k\uparrow$; $J'\downarrow \Rightarrow \frac{x}{k} - (1-\delta)\downarrow$.

$C_{xk} = -\frac{x}{k^2} J''\left(\frac{x}{k}-(1-\delta)\right) \leq 0$ if $J'' \geq 0$.

$V_{k\alpha} \geq 0 \Rightarrow F_{x\alpha} \geq 0$.

$V^T_{k\alpha} \geq 0$ (need $\Pi_{k\alpha} \geq 0$) $\rightarrow$ gets us selection hysteresis

$V^{T-1}_{k\alpha} \geq 0$

if 3 ⊕ cross-partials will get

if $F_{kx} \geq 0$; $F_{k\alpha} \geq 0$, $F_{x\alpha} \geq 0$.

then [illegible] $\Rightarrow V_{k\alpha} \geq 0$.

Econ 609 (Lec) 28/01/2008.

$\gamma \rightarrow x.$

$V^t(k_t; \alpha, \beta, \gamma, z_t) = \max_x \Pi(k, x; z_t) - P(\gamma, t) C(x, k; z_t) + E_t D^{t+1}(\beta, z_{t+1}) V^{t+1}(x, \alpha, \beta, \gamma; z_{t+1})$

To show $F_{x\gamma} \leq 0$ (higher price lowers investment)

Assume $P_\gamma \geq 0$.

$F_{x\gamma} = -P_\gamma C_x + E_t D^{t+1}(\beta, z_{t+1}) V^{t+1}_{k\gamma}(x, \alpha, \beta, \gamma, z_{t+1})$

"Need" $V^{t+1}_{k\gamma} \leq 0$. i.e. V submodular in k, γ.

Assume $C_x \geq 0$ (costs something currently to ↑ investment tomorrow)

To get V supermodular in $k, -\gamma$, need F supermodular $k, -\gamma, x$.

translating: $x - (1-\delta)k + I$, $I = x - (1-\delta)k$.

$F^t_{k\gamma} \leq 0$; $F_{k\gamma} = -P_\gamma C_k \leq 0$ if $C_k \geq 0$

$F_{kx} \geq 0$ (need $C_{kx} \leq 0$)

Firm is building up their own capital - not renting capital

$F_{x\gamma} \leq 0$ (as long as $V^{t+1}_{k\gamma} \leq 0$ → can get this by induction)

$F_{x\gamma} = -P_\gamma C_x + E_t D^{t+1}(\beta, z_{t+1}) V^{t+1}_{k\gamma}(x, \alpha, \beta, \gamma, z_{t+1}) \leq 0$

$C_x \geq 0$ ✓ need $V^{t+1}_{k\gamma} \leq 0$.

What about a temporary investment tax credit: V^{t+1} has no γ.

$\Rightarrow F_{x\gamma} = -P_\gamma C_x \leq 0$. ✓

∴ temp. investment tax credit will stimulate investment

but the longer lived, the less of an impact you might get

Characteristics of a steady state, and the fact that you are heading toward one, gives the problem a lot of structure.

We are doing things in more generality - regardless of steady state etc.

Let $D_\beta \geq 0$.

$F_{x\beta} = E_t D^{t+1}_\beta(\beta, z_{t+1}) V^{t+1}_k(x, \alpha, \beta, \gamma; z_{t+1}) + D^{t+1} V^{t+1}_{k\beta} \geq 0$

$F_{x\beta} = E_t D^{t+1}(\beta, z_{t+1}) V_k$

Lower IR. $F_{x\delta} = E_t D_\beta D^{t+1}(\beta, z_{t+1}) V_k + \ldots$ → recursive proof.

Want $F_{x\beta} \geq 0$, need $V^{t+1}_{k\beta} \geq 0$ (2), $V_k \geq 0$ (1).

(1) $V_k \geq 0$: $V(k, \ldots) = \max_x F(k, x, \ldots)$

$V(k+\delta) - V(k) = \max_x F(k+\delta, x) - \max_x F(k, x)$

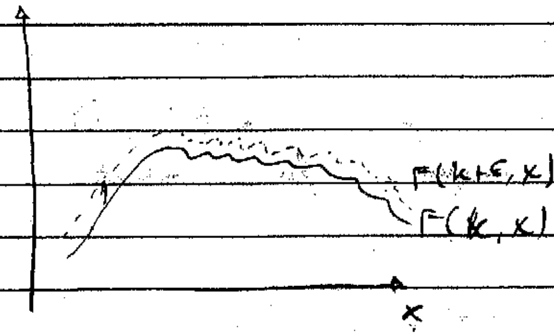
$\geq F(k+\delta, x^*) - F(k, x^*)$ w/ $x^* \in \arg\max_x F(k, x)$

(by sloppy) ≥ 0 if $F(\cdot)$ ↑ in k. PTO →

$V^{t+1}_{k\beta}$

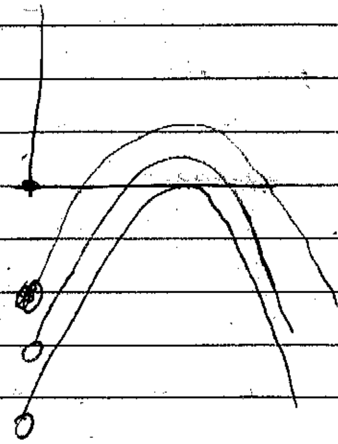
V_k

Coupled value via Envelope Thm.



If you push up objective f^n, then the max must have gone up.

True even if not differentiable:



② $V_{k\beta} \geq 0$.

need $F_{k\beta} \geq 0$:

$F_{k\beta} = 0$ ✓.

[$F_{kx} \geq 0$ (need $C_{kx} < 0$ by previous argument)].

$F_{x\beta} \geq 0$ — need $V_{x\beta} \geq 0$ ✓

$V_{k\beta}$ true at last

$V^{t+1 \geq 0}_{k\beta} \Rightarrow \underline{F_{x\beta} \geq 0}$; $\underset{\text{trivial}}{F_{k\beta} = 0}$, $F_{kx} \geq 0$.

By supermodularity thm, $\underline{V^{t}_{k\beta}} \geq 0$. ✓

And the argument holds much.

Back to γ: Putting more structure on the problem

Still want to show that $\frac{\partial x}{\partial \gamma} \leq 0$.

Need more structure

$p^{t+1}(\gamma) = B^t(\gamma)\, p^t(\gamma)$.

Mean reverting $\quad P_\gamma \geq 0$

$B_\gamma \leq 0$.

Define $\mathcal{V}^t(k_t, \gamma, z_t) = \dfrac{V^t(k_t, \gamma, z_t)}{p^t(\gamma)}$

Divide by $p^t(\gamma)$

$$\text{then } \mathcal{V}^t(k_t, \gamma, z_t) = \max_x \frac{\Pi(k, \alpha; z_t)}{p^t(\gamma)} - C(x, k; z_t)$$

$$+ E_t\, D^{t+1}(\beta, z_{t+1}) \underbrace{\frac{V^{t+1}(x, \alpha, \beta, \gamma; z_{t+1})}{p^{t+1}(\gamma)}}_{\mathcal{V}^{t+1}(x, \gamma, z_{t+1})} \underbrace{\frac{p^{t+1}(\gamma)}{p^t(\gamma)}}_{B^t(\gamma)}$$

- pushing the x, β into background.

$$\mathcal{V}^t(k_t, \gamma; z_t) = \max_x \underbrace{\frac{\pi(k, z_t)}{p^t(\gamma)} - C(x, k; z_t) + E_t D^{t+1}(z_{t+1})\, \mathcal{V}^{t+1}(x, \gamma; z_{t+1})\, B^t(\gamma)}_{\mathcal{L}}$$

$$\mathcal{L}_{x\gamma} = E_t D^{t+1}\Big[\underbrace{\mathcal{V}^{t+1}_{x\gamma} B^t(\gamma)}_{\text{gotta do the recursion here.}} + \underbrace{\mathcal{V}^{t+1}_k(x, k; z_{t+1})}_{\ge 0}\, \underbrace{B^t_\gamma(\gamma)}_{\le 0}\Big] \overset{?}{\le} 0$$

(weak) linear in ... of the shock to P)

↳ NOT pleased w/

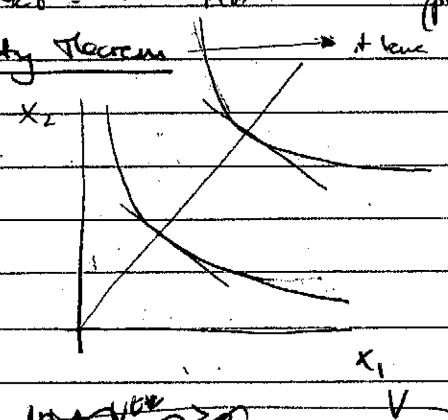
Supermodularity in $k, -\gamma, x$.

$\mathcal{L}_{kx} \ge 0$ ($C_{xk} \le 0$) ✓

$$\mathcal{L}_{k\gamma} = \frac{\partial}{\partial \gamma} \frac{\pi_k(k, \cdot)}{p(\gamma)} = -\frac{\overset{\ge 0}{p'_\gamma}}{(p^t)^2}\pi_k \le 0 \qquad \pi_k \ge 0 \checkmark$$

Symmetry Theorem → if have some sort of symmetry to the maximization, will also have a corresponding symmetry of the maximizer

eg. Homotheticity.



~~Want $V^{t+1}_{k\beta} \ge 0$~~

~~Want $F_{k\beta} \ge 0$~~ Want $F_{k\beta} \ge 0$ → $\beta \uparrow X$.

→ $V_k \ge 0$ and $V^{t+1}_{k\beta} \ge 0$. by induction?

have one ✓ ~~need to show~~ $V^t_{k\beta} \ge 0$

What other assumptions do need to

F supermodular:

$F_{k\beta} \ge 0$ ($= 0$ ✓)

$F_{kx} \ge 0$ ($= -p\, C_{xx}$)

$F_{k\beta} \ge 0$

~~$F_{x\beta} \ge 0$~~

→ $F_{kx} \ge 0$

$\Rightarrow V^t_{k\beta} \ge 0$ by Lever-Max Supermodularity.

this proves $V^{t+1}_{k\beta} \ge 0$, and treat it true being an assumption.

– paying (Arguax continuous?) ① 610 – cont time Stochastic optimization

Instant Advice (it excess return ⊛): when young – put all money in stocks b/c most of your assets are in human capital.

ECON 609 (lec) 30/01/2008

Example: Bellman's Equation for a diffusion process (on k) ~ Brownian Motion

⊛ $$\rho V^t(k_t) - V_t^t(k_t) = \max_x U(k_t, x) + V_k^t(k_t) A(k_t, x) + V_{kk}^t(k_t)\Omega(k_t, x)$$

$$V_t^t = \frac{\partial V^t}{\partial t}$$

if you take h→0.

$$V_t = \max_x hU(k, x) + e^{-\rho h} E_t(V_{t+h})$$

↳ lifetime utility (not maximized)

Constraint: (evolution of k) $$k_{t+h} = k_t + hA(k_t, x_t) + \sqrt{h\,\Omega(k_t, x_t)} \cdot \tilde{\varepsilon}_{t+h}$$

↳ transition eqⁿ

$$\tilde{\varepsilon}_{t+h} = \begin{cases} 1 & \text{w/ prob. } \frac{1}{2} \\ -1 & \text{w/ prob. } \frac{1}{2} \end{cases}$$

$$Var_t(\tilde{\varepsilon}_{t+h}) = 1$$

$$E_t(\tilde{\varepsilon}_{t+h}) = 0.$$

This would give discrete time Bellman eqⁿ:

$$V^t(k_t) = \max_x hU(k_t, x) + e^{-\rho h} E_t V^{t+1}(k_t + hA + \sqrt{h\Omega} \cdot \tilde{\varepsilon}_{t+h})$$ ⟩ take the limit h→0 to get ⊛.

↳ get everything on one side
- divide by h.
- use L'Hopital's Rule (x2)

Aside:

Sequence of 2 pt. risks ($\tilde{\varepsilon}_{t+h}$) can span any diffusion

↳ easier to span a diffusion: eg. Black-Scholes: just need two assets to span diffusion (eg. Treasury Bill + risky asset)

As long as you're spanning the space, our regular proof technique will work.

Kimball's own defⁿ: Publicly Complete Markets: whenever there is a tradable risky asset, there is also a full set of options on that asset (or the equivalent).

proving things in cont-time → prove in discrete time w/ 2-point risk and then take the limit

Merton Problem: Individual Saving for retirement – how many stocks to hold

$$\max_{c, \alpha} E_0 \int_0^\infty e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} dt$$

consumption (c), wealth in stock (α)

intertemporal constraint:

k_0 fixed – amt. of wealth $$dk = (\underbrace{rk + \alpha\mu - c}_{A})dt + \alpha\sigma dz$$ ← mean growth rate of wealth

$\alpha \in [0, k]$ – contemporaneous constraint

Closed form solutions in stochastic problems are only possible if you have powerful symmetries

contemporaneous w/ μ = expected return on stock $- r$.

$$dz = \underbrace{\sqrt{dt}}_{\sqrt{h}}\tilde{\varepsilon}$$, σ – std. dev per Γ of unit time

μ – hard to pin down; σ – easy to pin down] empirically

PTO →

$\int e^{-rt} u$

Price · Anomalies: high-stock mkt over-valued.

$\mathcal{H} = u + \lambda[\dot{\ }]$

$\frac{\partial \mathcal{H}}{\partial c} = 0; \quad \frac{\partial \mathcal{H}}{\partial k} = -\dot\lambda$

$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot k$

need to transformation to NOT disturb constraints

↳ Both constraints still need to be happy.

– transformation must have a simple effect on lifetime utility f^n

⇒ transformation will have the same effect on the Value f^n

not here, but it can happen

Capitalization Symmetry

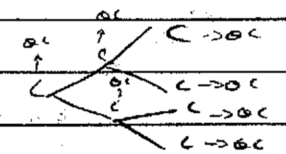
Transformation: $k \to \theta k$

$c \to \theta c$

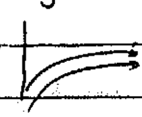
$\alpha \to \theta \alpha$

$\nu \to \theta^{1-\gamma} \nu$ → will carry nicely through $E(\cdot)$

↳ all c's got multiplied by θ.



$[\gamma > 1$



→ utility values in half is good

Does it respect the constraints?

- $\alpha \in [0, k] \iff \theta\alpha \in [0, \alpha k]$ ✓

↳ have to have ## (if only we may would be in Reser Max load)

- $dk = (rk + \alpha\mu - c)\,dt + \alpha\sigma dz$

$\Updownarrow$

$d(\theta k) = (r(\theta k) + (\theta\alpha)\mu - \theta c)\,dt + (\theta\alpha)\sigma dz$ ✓

$= \theta dk$ b/c θ is constant

f .... Then $V^t(\theta k) = \theta^{1-\gamma} V^t(k)$. in the dimension of k we know what value function looks like

To see the shape of $V(k)$, just choose $\theta = \frac{1}{k}$

$V^t\left(\frac{1}{k}k\right) = \left(\frac{1}{k}\right)^{1-\gamma} V^t(k)$

$\boxed{V^t(k) = k^{(1-\gamma)} V^t(1)}$

Capitalization Symmetry

Change: $dk = (rk + \alpha\mu + \underset{\text{non-capital income}}{y} - c) + \alpha\sigma dz$

Capitalizing

$$\int_0^T e^{-rt} y\,dt = y\left[\frac{e^{-rt}}{-r}\right]_0^T = y\left(\frac{e^{-rT}-1}{-r}\right) = \frac{y}{r}\left(1 - e^{-rT}\right)$$

Transformation: $y \to y - \theta$

$k \to k + \frac{\theta}{r}\left(1 - e^{-r(T-t)}\right)$

$(c \to c)$

$\nu \to \nu$

$\alpha \to \alpha$

$dk + d\left(\frac{\theta}{r} - \frac{\theta}{r} e^{-r(T-t)}\right) dt$

$= dk + 0 - r \frac{\theta}{r} e^{-r(T-t)}$

w/ liquidity constraints things get tricky.

$$k \geq 0 \quad \cancel{\Rightarrow} \quad k + \frac{\theta}{r}(1 - e^{-r(T-t)}) \geq 0.$$

∴ Assume PIH (no liquidity constraints)

$$d\left[k + \frac{\theta}{r}(1 - e^{-r(T-t)})\right] \stackrel{?}{=} \left[r \cdot \left(k + \frac{\theta}{r}(1 - e^{-r(T-t)})\right) + \alpha\mu + y - \theta - c\right] dt$$

$$dk - \frac{\theta}{r} r e^{-r(T-t)} = \cdots$$

and it goes both ways.

then $V\left(k + \frac{\theta}{r}(1 - e^{-rT}), y - \theta\right) = V(k, y)$

$y = \theta$ $\quad V\left(\underline{k + \frac{\theta}{r}(1 - e^{-rT})}, 0\right) = V(k, y)$

visible wealth + lifetime PV of inc.

Econ 609 (lec) 04/02/2008

Result from last time
$$V^t(k; z_t) = k^{1-\gamma} V^t(1; z_t)$$
Substitute into Bellman eq^n k will drop out - if it doesn't cancel out, you have made a mistake somewhere
→ DO IT

FOC - pretty easy

Proof of the symmetry th^m

	k	x
$\hat{k}$	✓	0
$\hat{x}$	✓	✓

IF

(a) Symmetry of the contemporaneous constraints

V = lifetime utility

V - value f^n.

$$(k, x) \in \mathcal{W} \iff (\overset{\hat{k}}{T_k(k)}, \overset{\hat{x}}{T_x(k, x)}) \in \mathcal{W}$$
includes z, parameters — state transformation — control var. transformation

(b) Symmetry of the intertemporal constraints (transition function)
$$\Gamma(T_k(k), T_x(k,x), \tilde{\omega}_{t+h}) = T_k(\Gamma(k, x, \tilde{\omega}_{t+h}))$$
→ for the same realization of the RV, gamma of the transformation equals the transformation of the the gamma

w/ $k_{t+1} = \Gamma(k_t, \tilde{\omega}_{t+h})$ $\quad \Gamma(\hat{k}, \hat{x}) = \underbrace{T_k(\Gamma(k, x))}_{k_{t+1}}$

(c) Symmetry of lifetime associated w/ the transformation T
$$S(\nu_t, k_t) = \Psi\{T_k(k_t), T_x(k_t, x),$$
translated lifetime utility
$$- E_t[S(\nu_{t+h}, \Gamma(k_t, X, \tilde{\omega}_{t+h}))]\}$$

$E_t \int_t^T e^{-\rho(\tau-t)} \ln C \, d\tau$ $\quad \begin{cases} k \to \theta k \\ x \to \theta k \\ c \to \theta c \end{cases}$

$\to E_t \int_t^T e^{-\rho(\tau-t)} (\ln\theta + \ln C) d\tau$

then
$$V^t(T_k(k)) = S(V(k), k) \quad \forall (k, x) \in \mathcal{W}$$
$= E_t \int_t^T e^{-\rho(\tau-t)} \ln c \, d\tau + \ln\theta \int_t^T e^{-\rho(\tau-t)} d$

{possible 4^th condition (d) $\nu_{T+h} \equiv 0$} $\quad$ depends on k_t or $\frac{1-e^{-\rho(T-t)}}{\rho}$

note: $\nu_t = \Psi(k_t, x, E\nu_{t+h})$ → actually $S(\Psi^T(k_T, x_T), k_T)$
$$= \Psi^T(T_k(k_T), T_x(k_T, x_T))$$
w/ $\nu_T = \Psi^T(k_T, x_T)$

→ at T or $T+h$, T on the inside does the same as S on the outside

Backward induction - it works at T, we then can iterate it back, satisfying the constraints the result at any t will be the same

PTO →

Kimball ride (SLOSH) old website

Symmetry Thm. ②

past website.

Assume $S \uparrow$ in V.

Proof strategy:

(i) Show $V(T_k(k)) - S(V(k), k) \geq 0$

(ii) show that $V(T_k(k)) - S(V(k), k) \leq 0$.

$$\text{(ii)} \Leftrightarrow S(V(k), k) - V(T_k(k)) \geq 0. \qquad (\forall k)$$

$$S(V(T_k^{-1}(k)), T_k^{-1}(k)) - V(k) \geq 0.$$

Application

Aside: Eg. of symmetry Thm:

$$V^t(\Theta k, T) = \frac{1 - e^{-\rho(t-T)}}{\rho} \ln \Theta + V(k, T)$$

In particular, if $\Theta = \frac{1}{k}$: $V^t(1, T) = \frac{1 - e^{-\rho(t-T)}}{\rho}(-\ln k) + V^t(k, T)$

$$\Rightarrow V^t(k, T) = V^t(1, T) + \ln k \left[\frac{1 - e^{-\rho(t-T)}}{\rho}\right]$$

Proof of (ii)

$S(V(T_k^{-1}(k)), T_k^{-1}(k)) - V(k)$

$$= S\left\{\max_{\hat{x}} \Psi(T_k^{-1}(k), \hat{x}, E_t V^{t+h}[\Gamma(T_k^{-1}(k), \hat{x}, \tilde{\omega}_{t+h})]), T_k^{-1}(k)\right\}$$

$$- \max_x \Psi\{k, x, E_t V^{t+h}[\Gamma(k, x, \tilde{\omega}_{t+h})]\}$$

x^* argmax of the 2nd one

slopping choice

$$\geq S\{\Psi(T_k^{-1}(k), T_x^{-1}(k, x^*), E_t[V^{t+h}(\Gamma[T_k^{-1}(k), T_x^{-1}(x^*, k), \tilde{\omega}_{t+h}])], T_k^{-1}(k)\}$$

$$- \Psi\{k, x^*, E_t V^{t+h}[\Gamma(k, x^*, \tilde{\omega}_{t+h})]\}$$

$V_k^t(k)$

Comma can interchange w/ transform

could be $=$

$$\geq S\{\Psi(T_k^{-1}(k), T_x^{-1}(k, x^*), E_t[V^{t+h}(T_k^{-1}(\Gamma(k, x^*, \tilde{\omega}_{t+h})))], T_k^{-1}(k)\}$$

$$- V^t(k).$$

S on the outside, slide T on the inside

$$\geq \Psi\{k, x^*, E_t S V^{t+h}[T_k^{-1}\Gamma(k, x^*, \tilde{\omega}_{t+h})], T_k^{-1}(k)\}$$

$$- \Psi\{k, x^*, E_t V^{t+h}(\Gamma(k, x^*, \tilde{\omega}_{t+h}))\}$$

by induction $SV^{t+h}(\ldots) - V^{t+h} \geq 0$

≥ 0 by (i)

Need $\Psi \uparrow$ in 3rd argument

~~(by ...)~~

(1)

respects weak first order stochastic dominance
$\Omega(F^2)$ FOSD $\Omega(F^1)$

Econ 609 (lec) eg. E_t 08/02/2008.

Proof of (ii)

$$V(k) = \max_{x \text{ s.t. } (k,x)\in W} \Psi(k, x, \Omega_t(V^{t+h}(\Gamma(k, x, \tilde{\omega}_{t+h}))))$$

$$= \Psi(k, x^*, \Omega_t(V^{t+h}(\Gamma(k_t, x^*, \tilde{\omega}_{t+h}))))$$

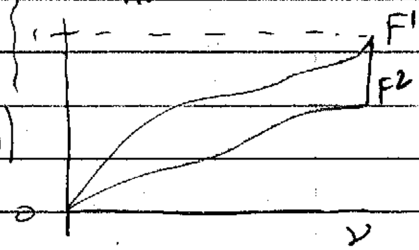
induction hypothesis $= \Psi(k, x^*, \Omega_t(S(V^{t+h}(T_k^{-1}(\underbrace{\Gamma(k, x^*, \tilde{\omega}_{t+h})}_{k_{t+h}}))), T_k^{-1}(k))$

$F^2_{V_{t+h}}(v) \leq F^1_{V_{t+h}}(v)$ $\Rightarrow F^2$ FOSD F^1

next period (t+h): S on the outside of V has the same effect as T on the inside

$v_t = \Psi(k_t, x_t, \Omega_t(\tilde{v}_{t+h}))$

Base case: $S^{T+h}(0, k) = 0$. $\hat{v}_{t+h} \equiv 0$.



symmetry of preferences preserved inter-temporally $= S(\Psi(T_k^{-1}(k), T_x^{-1}(k, x^*), \Omega_t($
$V^{t+h}(T_k^{-1}(\Gamma(k, x^*, \tilde{\omega}_{t+h}))), T_k^{-1}(k)))$

symmetry respects intertemporal BC (transition eq.) $= S(\Psi(T_k^{-1}(k), T_x^{-1}(k, x^*), \Omega_t(V^{t+h}$
$(\Gamma(T_k^{-1}(k), T_x^{-1}(k, x^*), \tilde{\omega}_{t+h})))), T_k^{-1}(k))$

$AB = BA$ $\quad A^{-1}AB A^{-1} = A^{-1}BAA^{-1}$
$\Rightarrow BA^{-1} = A^{-1}B$

transforming at date t changes things at date t+h

Functions can be thought of matrices

prove Max $\leq S[\max_{\hat{x} \text{ s.t. } (T(k), \hat{x})\in W} (\Psi(T_k^{-1}(k), \hat{x}, \Omega_t(V^{t+h}(\Gamma(T_k^{-1}(k), \hat{x}, \tilde{\omega}_{t+h}))), T_k^{-1}(k)]$

symmetry of contemporaneous constraints and s.t. if $(k, x) \in W$ then $(T_k^{-1}(k), T_x^{-1}(k, x)) \in W$

$= S(V^t(T_k^{-1}(k)), T_k^{-1}(k))$

$\Rightarrow S(V^t(T_k^{-1}(k)), T_k^{-1}(k)) - V(k) \geq 0$

Proof of (i)

$$S(V^t(k_t), k_t) = S(\max_{x \text{ s.t. } (k,x)\in W} \Psi(k_t, x, \Omega_t V^{t+h}(\Gamma(k_t, x, \tilde{\omega}_{t+h}))), k_t)$$

$$\overset{\text{def}^n \text{ of } x^*}{=} S(\Psi(k_t, x^*, \Omega_t V^{t+h}(\Gamma(k_t, x^*, \tilde{\omega}_{t+h}))), k_t)$$

S on the outside = T on the inside

symmetry of preferences preserved intertemporally $= \Psi(T_k(k), T_x(k, x^*), \Omega_t S[V^{t+h}(\Gamma(k_t, x^*, \tilde{\omega}_{t+h})), \Gamma(k_t, x^*, \tilde{\omega}_{t+h})]$

$\hookrightarrow$ S in the future conditional on the variable the future.

induction hypothesis $= \Psi(T_k(k), T_x(k, x^*), \Omega_t[V^{t+h}(T_k(\Gamma(k_t, x^*, \tilde{\omega}_{t+h})))])$

PTO $\rightarrow$

Symmetry of the transition f'n $\psi\left[T_k(k), T_x(k_t, x^t), \Omega_t\left(V^{t+h}\left(\Gamma\left(T_k(k_t), T_x(k_t, x^t), \tilde{\omega}^{t+h}\right)\right)\right)\right]$

Presev Max principle All symmetries of contemporaneous constraints $\leq \max_{\hat{x} \text{ s.t. } (T_k(k), \hat{x}) \in \mathcal{W}} \psi\left[T_k(k), \hat{x}, \Omega_t\left(V^{t+h}\left(\Gamma\left(T_k(k_t), \hat{x}, \tilde{\omega}^{t+h}\right)\right)\right)\right]$

$= V^t(T_k(k))$

Since $V(k) \leq S(V(T_k^{-1}(k)), T_k^{-1}(k))$ $\forall k$ includes $T_k(k)$

$\therefore V(T(k)) \leq S(V(k), k)$

and we have $S(V(k), k) \leq V(T_k(k))$

so $S(V(k), k) = V(T_k(k))$

Example $E_0 \int_0^\infty \frac{e^{-\rho t} e^{-aC}}{-a}\, dt$

$C \to C + \theta$

$W \to W + \theta P$ real case

↳ θ is a consumption per year, have price of

$\nu \to e^{-a\theta} \nu$

then $V(W + \theta P, z_t) = e^{-a\theta} V(W, z_t)$

$\theta = -\frac{W}{P}$ $\Rightarrow$ $V(0, z_t) = e^{+\frac{aW}{P}} V(W, z_t)$

$V(W, z_t) = e^{-\frac{aW}{P}} V(0, z_t)$

More averse low P

less averse at high P

→ concavity used for differentiability (1) 11/02/2008
which are ruled out by: changing the old $f(k,\rho)$
→ anything in b/w must be smooth

ECON 604 (Lec)

→ "published" solutions

Examples of Symmetry Th^m - an old website

Types of Symmetries:

① Scale Symmetries

② Capitalization Symmetries

③ Time Invariance → very intuitive

④ Rate of Time Symmetry → "almost nobody knows this one"

⑤ Additive Symmetry (exponential utility) → Markow + Kimball - precautionary savings and the timing of taxes.

$\Rightarrow V^t(\theta k) = a(\theta,\rho) + b(\theta,\rho)V(k)$

Set $\theta = \frac{1}{k}$

① $k \to \theta k$
$x \to \rho x$ } ~~Scale~~ if k, x are vectors, could have matrix scale

$V \to a(\theta,\rho) + b(\theta,\rho)V$ → need linear in V, but $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ could be very non-linear $f^{n's}$

$V(\hat{k}) = V(k)$

② $\binom{k}{x} \to \binom{\hat{k}}{\hat{x}}$ } can take very generally b/c
$V \to V$ → leaving this unchanged } $c \to c$, so $u(c)$ unchanged and so works w/ any preferences

→ Ricardian Equivalence → don't change BC

Merton - Miller Th^m

③ $t \to t+\theta$ not changing anything, just pushing
$T \to T+\theta$ things ~~later~~ forward - delaying things
$k \to k$
$x \to x$ that is $\hat{X}_{t+\theta} = X_t$; $\hat{k}_{t+\theta} = k_t$

$V^{t+\theta}(k, T+\theta) = V^t(k,T)$

let $\theta = -t$ ∴ $V^0(k, T-t) = V^t(k,T)$

if $T = \infty$ $V^0(k,\infty) = V^t(k,\infty)$ → doesn't depend on time

④ $\rho \to \theta\rho$
$r \to \theta r$ $\hat{C}(\tau) = \theta C(t, \frac{\tau - t}{\theta})$
$p \to \theta p$ $h \to \frac{h}{\theta}$

PTO →

→ annual basis vs. quarterly basis

"Making everything happen faster"

↳ can't change results b/c you're just changing how you measure things

↳ take everything that is per unit time and double it.

$$\hat{\alpha}(t) = \alpha\left(t + \frac{\tau - t}{a}\right)$$

$$\hat{k}(t) = k\left(t + \frac{\tau - t}{a}\right)$$

$v \to \hat{v}$ → different thing could happen b/c you're changing c, and h

↳ complicated model might take a lot of adjustments

there always exist in a

dynamic problem - ~~reducing~~ reduces the dimensionality of the parameter space

Other basic types of Structure

· Additive Separability (Modularity)

$$\max_{x_1, x_2} F_1(k_1, x_1) + F_2(k_2, x_2) = \max_{x_1} F_1(k_1, x_1) + \max_{x_2} F_2(k_2, x_2)$$

· Optimization Subproblem

$$V = \max_{M} \Pi(k, M) - C(k, x) + EV(x)$$

↳ choosing optimal M - no trade off - just choose the best

· Euler Equations → spend 1\$ less today

spend \$ e^{rh} more, h time from now

} like transversality ↳ obeys B.C. ↳ symmetry of B.C's

not quite symmetry theorem: messing up value $F^{\underline{n}}$

$$-U_{ct} + E_t e^{rh} U_{ct+h} \leq 0.$$

↳ if you started out optimizing, cannot be better off w/ this change → it might or might not

Then, w/ Euler Eq$^{\underline{n}}$'s: $\hat{k}_t = k_t$ but $\hat{k}_{t+h} \stackrel{\neq}{=} k_{t+h}$

↳ leave starting the state var the same

$\hat{X}_t$

→ in the distant future, need state and control vars back to square one in a finite amount of time → after this amt of time need to get habit back and touch b/c $u(c)$.

Habit Formation $U(C, H)$; $\dot{H} = \rho(C - H)$

↳ a complicated $F^{\underline{n}}$ of H

Δ'ing consumption, change habit, but then you can revert it

"Euler Eq^n"

Example: $U(C, H)$

(discrete time) $H_{t+1} = H_t + \xi(C_t - H_t)$ ⟶ consumption is the only thing your control

constant

state $= (1-\xi)H_t + \xi C_t$

Spend ε less today; spend $e^r\varepsilon$ more: put consumption back on track.

⟶ changed habit ⟶ $\hat{C}_t = C_t + \varepsilon$

$\hat{H}_{t+1} = H_{t+1} + \xi\varepsilon$ $\qquad \hat{C}_{t+1} = C_{t+1} - \varepsilon e^r$

Wealth back on track $\begin{cases} \hat{W}_{t+1} = W_{t+1} + \varepsilon e^r \\ \hat{W}_{t+2} = W_{t+2} \end{cases}$ (b/c I spent the extra money)

$$\hat{H}_{t+2} = (1-\xi)\underbrace{(H_{t+1} + \xi\varepsilon)}_{\hat{H}_{t+1}} + \xi[\underbrace{C_{t+1} - \varepsilon e^r}_{\hat{C}_{t+1}}]$$

$\xi < 1$

$$= H_{t+2} + (1-\xi)\xi\varepsilon - \xi\varepsilon e^r$$ ⟶ have a smaller habit — end up spending less and more result

~~If we do nothing~~

$$= H_{t+2} + \varepsilon\xi[e^r - (1-\xi)]$$

If we do nothing $\hat{C}_{t+2} = C_{t+2}$, $\hat{C}_{t+3} = C_{t+3}$

$$\hat{H}_{t+3} = H_{t+3} - (1-\xi)\varepsilon\xi[e^r - (1-\xi)]$$
$$\hat{H}_{t+4} = H_{t+4} - (1-\xi)^2\varepsilon\xi[e^r - (1-\xi)]$$

So, change consumption:

$$\begin{cases} \hat{C}_t = C_t + \varepsilon \\ \hat{C}_{t+1} = C_{t+1} - \varepsilon e^r - (1-\xi)\varepsilon \\ \hat{C}_{t+2} = C_{t+2} + (1-\xi)\varepsilon e^r \\ \hat{C}_{t+3} = C_{t+3} \end{cases}$$

in wealth terms cancel out, but reverses the habit

Wealth will have some complicated track, but $\hat{W}_{t+3} = W_{t+3}$, and now we also get $\hat{H}_{t+3} = H_{t+3}$ — habit back on track

⟶ this gets easier w/ more control vars

Euler Eq^n: $E_t[U_{c_t}\xi - \xi[e^r + (1-\xi)]U_{c_{t+1}} + \xi(1-\xi)e^r U_{c_{t+2}}] = 0$.

PTO ⟶

ECON 609 (lec) 13/02/2008

Error in Habit formation: forgot utility discount factor

P.Set Qn #1: $\max \sum_{t=0}^{\infty} \beta^t U(C_t, H_t)$ ← habit

s.t. $W_{t+1} = R_t(W_t - C_t)$

↳ gross real return.

$H_{t+1} = \rho C_t + (1-\rho) H_t$ — getting states back on track.

Find and demonstrate an Euler eq^n with a finite number of terms

↳ different than in control theory.

P.Set Qn #2: In the same context where I prove that $RRA \geq \gamma$ is preserved, prove that relative prudence $\left(\frac{-K V_{KKK}}{V_{KK}} \geq \theta\right)$ is preserved (assume 1 period utility; value f^n inherits it).

weak separability · inner functions

$$\left\{ \begin{array}{l} \max F(\phi_1(U, \phi_2(H, E)), \ldots) \\ \bar{Y} = \phi_3(K, L, \phi_4(U, \phi_5(H, E))) \end{array} \right. \quad (*)$$

(U: work week of capital; E: effort; H: hours; …: other variables)

⇒ Optimization subproblem:

Necessary condition to solving (*).

$$\left\{ \begin{array}{l} \max\limits_{H,E} \phi_2(H, E) \\ \text{s.t. } \phi_5(H, E) = \text{constant} \end{array} \right.$$

(or min depending on which max F(.))

→ doesn't affect the constraint

for different values get different H^*, E^* → can map out for different constant values: $E(H)$

↳ need to do this to solve (*): max ϕ_2 st. constraint is not changed.

You find that E and H co-move → unobserved effort is a function of the hours

Optimization subproblems give you insight into the original problem.

Another one: $\min \phi_1(U, \phi_2(H, E))$

st. $\phi_4(U, \phi_5(H, E)) = \text{constant}$

→ single line: $U, H, E \Rightarrow U(H)$

Weak Separability: b/w (H,E) and U $\quad \frac{\partial}{\partial U}\left(\frac{F_E}{F_H}\right) = 0.$

→ if one var. appears by itself — just max out over it and can almost get rid of it.

PTO →

Island Analogy

It was important that (H, E) only appears in two places – not in other parts of the problem, saving into good state security

$$V^t(k_t) = \max_{S_1, S_2, \ldots, S_M} \left\{ U(k - \sum_i S_i) + \beta \sum_i \left(\Pi_i E_{t,(i)} \left[V^{t+1}(R_i S_i + \tilde{q}_i)\right]\right) \right\}$$

prob. of state i happening

conditional on state i

in cont. time becomes diffusion

M state scenarios ↳ good & bad states

Precautionary Saving Problem

Idiosyncratic Risk – risk apart from what stocks are doing.

Capitalization (Equivalency) ↓ Capitalized their labor income $(k + \tilde{q}_i)$

Write $\tilde{q}_i = \bar{q}_i + \tilde{\varepsilon}_i$, then if we re-write k appropriately, we can write:

$$V^t(k_t) = \max_{S_1, \ldots, S_M} \left\{ U(k - \sum_i S_i) + \beta \sum_i \left(\Pi_i E_{t,i} \left[V^{t+1}(R_i S_i + \tilde{\varepsilon}_i)\right]\right) \right\}$$

(Want to show that if $\frac{-c U_{cc}}{U_c} \geq 5$ then $\frac{-k V_{kk}}{V_k} \geq 5$ for the right k)

$\bar{q}_i$ can depend on stock market.

But, $\tilde{\varepsilon}_i$ has mean 0. – That's how we constructed it. It may or may not depend on what the stock market is doing – we will see later.

We specify the utility f^n: (If no ε's we would have perfect scale symmetry)

$$V^t(k_t) = \max_{S_1, \ldots, S_M} \frac{(k - \sum S_i)^{1-\gamma}}{1-\gamma} + \beta \sum \Pi_i E_{t,i} \left[V^{t+1}(R_i S_i + \varepsilon_i)\right] \qquad \frac{k^{1-\gamma}}{1-\gamma} \phi(\gamma)$$

then we would have $\frac{-k V_{kk}}{V_k} = \gamma$ (= RRA) if no $\tilde{\varepsilon}$ (idiosyncratic risk)

s.t. $\sum_i S_i + C = K$ (CONSTRAINT)

if idiosyncratic risk: RRA $\geq \gamma$

Conjugate F^n technique will work well w/ additive separability & linear BC

Want to show that background risk increases RRA in a multi-period world.

$$V^t(k_t) = \max U(\underbrace{k - \sum S_i}_{C}) + \beta \underbrace{\sum_i \Pi_i E_{t,i} \left[V^{t+1}(R_i S_i + \varepsilon_i)\right]}_{\Omega_i(S_i)} + \lambda\left[k - C - \sum S_i\right]$$

↳ Lagrangian

FOC: $u'(c) = \lambda$

need V, utility, Ω f^n concave

$$\beta \Pi_i E_{t,i} R_i V_k^{t+1}(R S_i + \tilde{\varepsilon}_i) = \lambda \qquad (\Omega_i'(S_i) = \lambda) \ \forall i$$

$$V_k^t(k) = \lambda \qquad \text{(Envelope Th}^m\text{)}$$

For concavity of $V(\cdot)$ you will usually need concavity of Ω and u.

$$\therefore\ V(k) - \lambda k = u(c) - \lambda c + \sum_i \Omega_i(s_i) - \lambda s_i$$

$$\underbrace{\max_k V(k) - \lambda k}_{-V^*(k)} = \underbrace{\max_c u(c) - \lambda c}_{-u^*(c)} + \underbrace{\sum_i \max_{s_i} \Omega_i(s_i) - \lambda s_i}_{\sum_i -\Omega_i^*(\lambda)}$$

Conjugate f^ns $\to$ λ?

$\therefore V^*(\lambda) = u^*(\lambda) + \sum_i \Omega_i^*(\lambda)$ (Conjugate Addition)

Conjugate f^n's add up!

Vertical addition: if V has property; Ω has property $\Rightarrow$ translate to conjugate f^n property – show that it adds nicely

$$V^*(\lambda) = \lambda k - V(k)$$
$$= \lambda V_k^{-1}(\lambda) - V(V_k^{-1}(\lambda)) \quad b/c \quad k = V_k^{-1}(\lambda)$$

(What property of V^* corresponds to V having RRA $\geq \gamma$.)

What is V^{**}?

$$V^*(\lambda) = \lambda V_k^{-1}(\lambda) - V(V_k^{-1}(\lambda)) \qquad \frac{dk}{d\lambda} = \frac{1}{V_{kk}(V_k^{-1}(\lambda))}$$

Inverse f^n Theorem

$$\therefore V^*_\lambda(\lambda) = V_k^{-1}(\lambda) + \underbrace{\cancel{\frac{(\lambda)}{V_{kk}(V_k^{-1}(\lambda))}}}_{\text{characteristic of inverse f}^n}$$

$V_k^{-1}(\lambda)$ $V_k(k)$

$$- \cancel{V_k(V_k^{-1}(\lambda)) \cdot \frac{1}{V_{kk}(V_k^{-1}(\lambda))}}$$

$$\therefore V^*_\lambda(\lambda) = k = V_k^{-1}(\lambda)$$

$V^*(\lambda) = \lambda k - V(k)$ $\to \lambda V_k^{-1}(\lambda) - V(V_k^{-1}(\lambda))$

} Symmetric

$\Rightarrow V(k) = \lambda k - V^*(\lambda)$ $\to V_\lambda^{*-1}(k) k - V^*(V_\lambda^{*-1}(k))$

$V^*_\lambda(\lambda) = k$
$\Rightarrow \lambda = V_\lambda^{*-1}(k)$

Main result $\to$ $V(k) = V^{**}(\lambda)$

Two columns in table: switch around V, V^* and change k, λ

Correlates V and V^*!

PTO $\to$

$$V^*_\lambda(\lambda) = k = V_k^{-1}(\lambda)$$

$$\therefore V^*_{\lambda\lambda}(\lambda) = \frac{1}{V_{kk}(V_k^{-1}(\lambda))} \qquad (5)$$

F. Set Q^c ⇒ What property does prudence look like in V^* - does this add up?

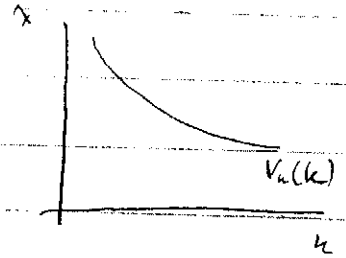
RRA Example:

$$\frac{-kV_{kk}}{k} \geq \gamma$$

Look at table: $k = -V^*_\lambda(\lambda)$; $V_{kk} = \frac{1}{V^*_{\lambda\lambda}}$; $V_k = \lambda$

$$\frac{-V^*_\lambda(\lambda)}{\lambda V^*_{\lambda\lambda}(\lambda)} \geq \gamma \qquad \text{assume } k > 0.$$

V has RRA $\geq \gamma$ ~~for V~~ ⇔ V^* has RRA $\leq \frac{1}{\gamma}$.



Elasticity going one way ⇒ $\frac{1}{\text{elasticity}}$ the other way (if you [illegible] matter)

$$\text{RRA} = -\frac{\partial \ln V_k}{\partial \ln k}$$

Example: Concavity $V_{kk} \leq 0$.

$$\Leftrightarrow \frac{1}{V^*_{\lambda\lambda}} \leq 0 \Rightarrow V^*_{\lambda\lambda} \leq 0.$$

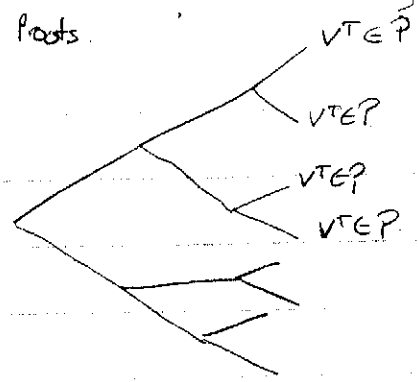
If the thing itself concave then conjugate f^n concave

Conjugate f^n Q^n

has same property?
18/02/2008

ECON 604 (lec)

Conjugate Roots



we want $V^{t+1} \in P$
$\Rightarrow V^t \in P$.

$V^t(k) = \max_{c,s_1,s_2} U(c) + \Omega_1(s_1) + \Omega_2(s_2)$ s.t. $c + s_1 + s_2 + \dots = k$

literature out here to help you - bolier, Christian

To prove $V^{t+1} \in P$ then $V^t \in P$. and $U_t \in P$.

(I) If $V^{t+1} \in P$ then $\Omega_t \in P$ where eg. $\Omega_t = \beta E_t V^{t+1}(\dots)$ for any concave f^n V, $V \in P \Leftrightarrow V^* \in P^*$ (so, this is for general f^n incl. u, Ω, V)

across states at vertical concavity expectation (Vertical aggregation / addition)

only have to prove this → (II) If $\Omega_t \in P \Leftrightarrow \Omega_t^* \in P^*$ → conjugate f^n having conjugate property

prove → Duality Table

(and also automatically $U_t \in P \Leftrightarrow U_t^* \in P^*$)

(III) If $\Omega_t^* \in P^*$ and $U_{t}^* \in P^*$ then $V_t^*(\lambda) = U^*(\lambda) + \sum_i \Omega_i^*(\lambda) \in P^*$

↳ Horizontal Addition

→ Conjugate F^{n}'s Add up easily - just addition } Simplifies the problem.
↳ Property P^* preserved under addition

• Need $V(\cdot)$ concave

Examples ① ~~~~ Concavity $V_{kk}(k) \le 0$

$\Leftrightarrow \frac{1}{V^*_{\lambda\lambda}(\lambda)} \le 0$,

which is $V^*_{\lambda\lambda}(\lambda) \le 0$ concavity

And (III) preserved under addition $\Rightarrow$ V concave:

② $\frac{kV_k}{V} \ge \gamma \Leftrightarrow \frac{V}{kV_k} \le \frac{1}{\gamma} \overset{\text{table}}{\Leftrightarrow} \frac{\lambda V^*_\lambda(\lambda) - V^*(\lambda)}{V^*_\lambda(\lambda)\lambda} \le \frac{1}{\gamma}$

$\Leftrightarrow 1 - \frac{V^*}{V^*_\lambda(\lambda)} \le \frac{1}{\gamma}$.

$\Leftrightarrow 1 - \frac{1}{\gamma} \le \frac{V^*(\lambda)}{\lambda V^*_\lambda(\lambda)}$

$\Leftrightarrow \frac{\gamma}{\gamma - 1} \ge \frac{\lambda V^*_\lambda(\lambda)}{V^*_\lambda(\lambda)}$

Now, all we need is this thing to add

PTO →

$$(1-\tfrac{1}{\gamma})\lambda V_\lambda^*(\lambda) \le V^*(\lambda)$$

will w. es $\ge \gamma$.

Addition problem $(1-\tfrac{1}{\gamma})\lambda u_\lambda^*(\lambda) \le u^*(\lambda)$

$$\oplus\quad \frac{(1-\tfrac{1}{\gamma})\lambda \Omega_\lambda^*(\lambda) \le \Omega^*(\lambda)}{(1-\tfrac{1}{\gamma})\lambda V_\lambda^*(\lambda) \le V^*(\lambda)}$$

Sum of derivatives is the derivative of sum

$\hookrightarrow$ if $u^* + \Omega^* = V^*$

then $u_\lambda^* + \Omega_\lambda^* = V_\lambda^*$

Same for $\frac{kV_k}{V} \le \gamma$.

$\to$ RRA.

(3) $\frac{-kV_{kk}}{V_k} \ge \gamma \iff \frac{-\lambda V_{\lambda\lambda}^*}{V_\lambda^*} \le \frac{1}{\gamma}$

$\cancel{-\lambda V_{\lambda\lambda}^* \le \frac{1}{\gamma}}$

$-\lambda u_{\lambda\lambda}^* \le \frac{1}{\gamma} u_\lambda^*$

Elasticities bigger than or less than a constant add nicely

$$\oplus\quad \frac{-\lambda \Omega_{\lambda\lambda}^* \le \frac{1}{\gamma}\Omega_\lambda^*}{-\lambda[u_{\lambda\lambda}^* + \Omega_{\lambda\lambda}^*] \le \frac{1}{\gamma}[u_\lambda^* + \Omega_\lambda^*]}$$

$$-\lambda V_{\lambda\lambda}^* \le \frac{1}{\gamma} V_\lambda^*$$

(4) $\frac{-kV_{kkk}}{V_{kk}} \ge 0 \iff \frac{-V_\lambda^*\left(\frac{-V_{\lambda\lambda\lambda}^*}{V_{\lambda\lambda}^{*3}}\right)}{\left(\frac{1}{V_{\lambda\lambda}^*}\right)} \ge 0.$

$$\iff \frac{V_\lambda^* V_{\lambda\lambda\lambda}^*}{V_{\lambda\lambda}^{*2}} \ge 0 \qquad (*)$$

Made up so that (+) definiteness of this matrix is (*)

~~Upper~~ Joint-concavity need neg. def. Hessian; here pos. def.

$$\begin{bmatrix} u_{\lambda\lambda\lambda}^* & \sqrt{\theta}\, u_{\lambda\lambda}^* \\ \sqrt{\theta}\, u_{\lambda\lambda}^* & u_\lambda^* \end{bmatrix}$$

$\to$ diagonal are pos (see 5) for $u_{\lambda\lambda\lambda}^* \ge 0$

Need main diagonal $\oplus$, and det $\oplus$ $\to$ assumed.

det $= u_{\lambda\lambda\lambda}^* u_\lambda^* - \theta\, u_{\lambda\lambda}^{*2} \ge 0 \to$ same as property (*)

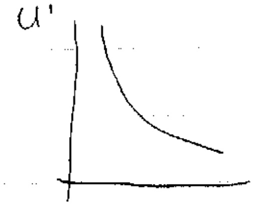
③

now it's obvious that this is preserved under addition b/c the sum of two pos. def matrices is pos. def.

this is how we got ... this → 3rd derivative ⊕ is the same as convex-ish having ⊕ 3rd derivative

5) . $U_{kkk} \geq 0$ $\iff \dfrac{-V_{\lambda\lambda\lambda}}{V_{\lambda\lambda}^{3}} \geq 0$ $\iff V_{\lambda\lambda\lambda} \geq 0$ $\leq$?

looking sideways



use this strategy when you have products

~~pos~~ pos def iff det ≥ 0 and ⊕ diagonals

neg. def iff det ≥ 0 and ⊖ diagonals

6) $\dfrac{V_{kkk} V_k}{V_{kk}^2} \geq 0$ (note similarity to conjugate property of u!)

and $\dfrac{u''' u'}{(u'')^2} = 0$ then you have concave consumption f^n

$\iff \dfrac{-\lambda u_{\lambda\lambda\lambda}}{u_{\lambda\lambda}} \geq 0$ and $\dfrac{-\lambda u_{\lambda\lambda\lambda}}{u_{\lambda\lambda}} = 0$

What about Ⓘ?

Show that if ~~λ~~ $\dfrac{-V_{kk}(k)}{V_k(k)} \geq \dfrac{\gamma}{k}$ then

$\hat{V}(k) = E_{\tilde{\varepsilon}}\, V(k+\tilde{\varepsilon})$ also has $\dfrac{-\hat{V}_{kk}(k)}{\hat{V}_k(k)} \geq \dfrac{\gamma}{k}$

Need $E(\tilde{\varepsilon}) = 0$

if $\dfrac{-V_{kk}(k)}{V_k(k)} = \dfrac{\gamma}{k}$ then $\dfrac{-V_{kk}(k+\tilde{\varepsilon})}{V_k(k+\tilde{\varepsilon})} = \dfrac{\gamma}{k+\tilde{\varepsilon}} < \dfrac{\gamma}{k}$

Define $M(\varepsilon) = \dfrac{V_k(k+\tilde{\varepsilon})}{E_{\tilde{x}} V_k(k+\tilde{x})}$ (MV of where you're at to average MV)

$A(\varepsilon) = \dfrac{-V_{kk}(k+\varepsilon)}{V_k(k+\varepsilon)}$

PTO →

recursive during
variable

$$-\frac{\hat{V}_{kk}(k)}{\hat{V}_k(k)} = \frac{-E_\varepsilon V_{kk}(k+\varepsilon)}{E_\varepsilon V_k(k+\varepsilon)} = \frac{\int_\varepsilon V_{kk}(k+\varepsilon)}{\int_x V_k(k+x)}$$

$$\times \frac{M(\varepsilon)}{M(\varepsilon)}$$

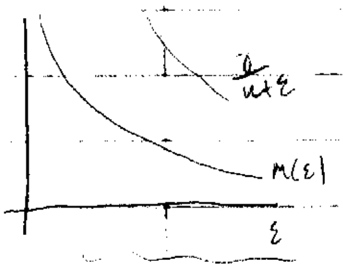
$$= E_\varepsilon \left[\frac{V_k(k+\varepsilon)}{E_x V_k(k+x)} \left(\frac{-V_{kk}(k+\varepsilon)}{V_k(k+\varepsilon)} \right) \right]$$

$$E_\varepsilon(M(\varepsilon)) = 1$$

$$= E_\varepsilon(M(\varepsilon)A(\varepsilon))$$

$$(?) \geq E_\varepsilon\left(M(\varepsilon)\frac{\gamma}{k+\varepsilon}\right)$$

$$= \underbrace{E_\varepsilon(M(\varepsilon))}_{1} E_\varepsilon\left(\frac{\gamma}{k+\varepsilon}\right) + Cov\left(M(\varepsilon), \frac{\gamma}{k+\varepsilon}\right)$$



$$\geq E_\varepsilon\left(\frac{\gamma}{k+\varepsilon}\right) = \gamma \frac{\ }{E(k+\varepsilon)} = \frac{\gamma}{k}$$

$$= \gamma E\left(\frac{1}{k+\varepsilon}\right) \geq \gamma \frac{1}{k} \quad \text{(Jensen's)}$$

$Cov(.,.) > 0$

Just add on the derivative for prudence

So now we've shown

$$\Omega_i(S_i) = \beta\pi_i E_\varepsilon V(R_iS_i + \tilde{\varepsilon})$$

$$\frac{E_\varepsilon V_{kk}(R_iS_i + \tilde{\varepsilon})}{E_\varepsilon V_k(R_iS_i + \tilde{\varepsilon})} \geq \frac{\gamma}{R_iS_i}$$

$$\Omega'(S_i) = \beta\pi_i R_i V_k(R_iS_i + \tilde{\varepsilon})$$

$$\Omega''(S_i) = \beta\pi_i R_i^2 V_{kk}(R_iS_i + \tilde{\varepsilon})$$

$$-\frac{S_i\,\Omega''(S_i)}{\Omega'(S_i)} = \frac{R_i^2}{R_i} S_i \frac{-E_\varepsilon V_{kk}}{E_\varepsilon V_k} \geq R_iS_i\frac{\gamma}{R_iS_i} = \gamma.$$

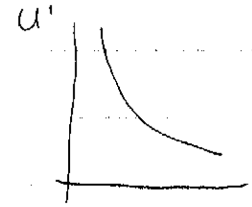
now its obvious that this is preserved under addition b/c the sum of two pos. def matrices is pos. def.

this is how we got at this
$\hookrightarrow$ 3rd derivative $\ominus$ is the same as conjugate having $\oplus$ 3rd derivative.

5) $V_{kkk} \geq 0 \iff \dfrac{-V_{\lambda\lambda\lambda}}{V_{\lambda\lambda}^{3}} \geq 0 \iff V_{\lambda\lambda\lambda} \geq 0$ $\leq$?

?

looking sideways



$\hookrightarrow$ use this strategy when you have products

pos def iff det ≥ 0 and $\oplus$ diagonals

neg. def iff det ≥ 0 and $\ominus$ diagonals

6) $\dfrac{V_{kkk} V_k}{V_{kk}^2} \geq 0$ (use similarity & conjugate property of u!)

and $\dfrac{u''' u'}{(u'')^2} = 0$ then you have concave consumption f^n

$\iff \dfrac{-\lambda u_{\lambda\lambda\lambda}}{u_{\lambda\lambda}} \geq 0$ and $\dfrac{-\lambda u_{\lambda\lambda\lambda}}{u_{\lambda\lambda}} = 0$

What about $\boxed{I}$?

Show that if $\dfrac{-V_{kk}(k)}{V_k(k)} \geq \dfrac{\gamma}{k}$ then

$\hat{V}(k) = E_{\tilde{\varepsilon}}\, V(k+\tilde{\varepsilon})$ also has $\dfrac{-\hat{V}_{kk}(k)}{\hat{V}_k(k)} \geq \dfrac{\gamma}{k}$

Need $E(\tilde{\varepsilon}) = 0$

if $\dfrac{-V_{kk}(k)}{V_k(k)} = \dfrac{\gamma}{k}$ then $\dfrac{-V_{kk}(k+\tilde{\varepsilon})}{V_k(k+\tilde{\varepsilon})} = \dfrac{\gamma}{k+\tilde{\varepsilon}} < \dfrac{\gamma}{k}$

Define $M(\varepsilon) = \dfrac{V_k(k+\tilde{\varepsilon})}{E_{\hat{x}} V_k(k+\hat{x})}$ (MV of where you're at to average MV)

$A(\varepsilon) = \dfrac{-V_{kk}(k+\varepsilon)}{V_k(k+\varepsilon)}$

PTO $\rightarrow$

ECOM 609 (lec) 20/02/2008

k is now a vector

$\lambda = V_k(k)$

$d\lambda = V_{kk}\, dk$ 1.2 on Final

$dk = [V_{kk}]^{-1} d\lambda$ (1)

$V_\lambda^* = k$

$V_{\lambda\lambda}^* d\lambda = dk$ (2)

(1) & (2) $\Rightarrow$ $V_{\lambda\lambda}^* d\lambda = [V_{kk}]^{-1} d\lambda$

$\therefore V_{\lambda\lambda}^* = [V_{kk}]^{-1}(V_k^{-1}(\lambda))$

$V_{\lambda\lambda}^* = V_{kk}^{-1}$

$dV_{\lambda\lambda}^* = V_{kk}^{-1}$

$V_{\lambda\lambda\lambda}^* = \frac{\partial}{\partial\lambda}[V_{kk}]^{-1}$

$d[V_{kk}]^{-1}$

$V_{kk}V_{kk}^{-1} = I$

$(dV_{kk})[V_{kk}]^{-1} + V_{kk}\, d[V_{kk}]^{-1} = 0$

$\therefore V_{kk}\, d[V_{kk}]^{-1} = -dV_{kk}[V_{kk}]^{-1}$

$\therefore d[V_{kk}]^{-1} = -[V_{kk}]^{-1} dV_{kk}[V_{kk}]^{-1}$

$$\frac{\partial [V_{kk}]^{-1}}{\partial k_i} = -[V_{kk}]^{-1}\frac{\partial V_{kk}}{\partial k_i}[V_{kk}]^{-1}$$

$$\sum_i -[V_{kk}]^{-1}\frac{\partial V_{kk}}{\partial k_i}[V_{kk}]^{-1} dk_i \quad (3)$$

(1) + (3) gets (1.2)

$$dk_i = \sum_j [V_{kk}]^{-1}_{ij}\, d\lambda_j$$

PTO →

②

Parameters : $V(k, p)$

$$\lambda = V_k(k, p) \iff k = V_k^{-1}(\lambda, p)$$

Strictly concave in k (or convex)

$$V^*(\lambda, p) \triangleq k V_k - V$$

$$= \underbrace{V_k^{-1}(\lambda, p)}_{k} V_k(\underbrace{V_k^{-1}(\lambda, p)}_{k}, p) - V(\underbrace{V_k^{-1}(\lambda, p)}_{k}, p) \quad (A)$$

$$\lambda = V_k(k, p) \Rightarrow d\lambda = V_{kk} dk + V_{kp} dp$$

$$\Rightarrow V_{kk} dk = d\lambda - V_{kp} dp$$

$$\therefore \frac{d\lambda}{V_{kk}} - \frac{V_{kp} dp}{V_{kk}} = dk \quad (*)$$

$k = V_k^{-1}(k, p)$

$$\therefore \frac{dk}{d\lambda} = \frac{1}{V_{kk}(V_k^{-1}(\lambda, p), p)}$$

$$\frac{\partial k}{\partial p} = \frac{-V_{kp}(V_k^{-1}(\lambda, p), p)}{V_{kk}(V_k^{-1}(\lambda, p), p)}$$ (holding λ fixed : how does k move w/ parameter)

$$(A) \to V_p^*(\lambda, p) = \frac{\partial}{\partial p}\Big|_{\lambda = \text{const}} (kV_k - V) \quad * \quad d[kV_k - V] = [V_k + kV_{kk} - V_k] dk + [kV_{kp} - V_p] dp$$

$$\overset{*}{=} kV_{kk} \underbrace{dk}_{\left(\frac{1}{V_{kk}} d\lambda - \frac{V_{kp}}{V_{kk}} dp\right)} + [kV_{kp} - V_p] dp$$

$$\overset{*}{=} \cancel{kV_{kk}}[$$

$$= k d\lambda - V_p dp \quad dV^*$$

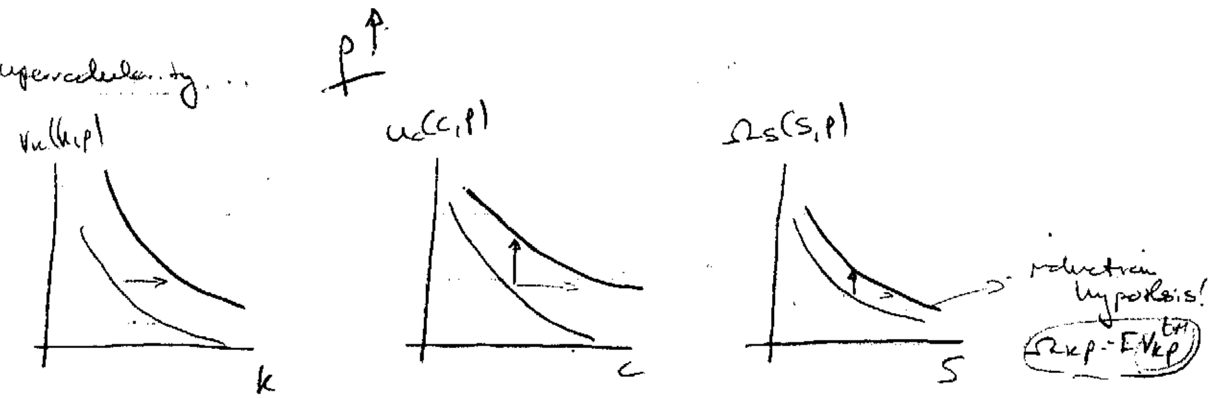
$$\therefore d(kV_k - V) = k d\lambda - V_p dp$$ } Same coefficients

And, from $d V^*(\lambda, p) = V_\lambda^*(\lambda, p) d\lambda + V_p^* dp$

General Principle. Two facts. $V_\lambda^*(\lambda, p) = k$ And $V_p^*(\lambda, p) = -V_p(k, p)$

w/ $k = V_k^{-1}(\lambda, p)$

Supermodularity...



Using the fact that upward shift is also a right ward shift.

∴ add up horizontally - shift right ...

But then shifting to the right, is shifting up as well.

Proof.

$V_{kp}^{t+1} \geq 0$ $\qquad$ $\Omega_{kp} \geq 0$ $\quad$ $U_{kp} \geq 0 \Rightarrow \Omega^*_{p\lambda} \geq 0$ $\cdot$ $U^*_{p\lambda} \geq 0$

$U_{kp} \geq 0$

I $\;$ If $V_{kp}^{t+1} \geq 0 \Rightarrow$ then $\Omega_{kp} - EV_{kp}^{t+1} \geq 0$ $\qquad$ (Vertical ✓)

II $\;\Big\{$ $V_p^* = -V_p$

$$V_{p\lambda}^* = \frac{-V_{pk}}{V_{kk}} \underset{\text{duality}}{\overset{\text{conjugate}}{\Longleftrightarrow}} V_{pk} = \frac{-V_{p\lambda}^*}{V_{\lambda\lambda}^*} \geq 0 \Rightarrow V_{p\lambda}^* \geq 0.$$

III $\;$ Add up: $V_{p\lambda}^* = \Omega_{p\lambda}^* + U_{p\lambda}^* \geq 0$ ✓

$\Rightarrow V_{pk} \geq 0$ #

b/c $V^* = \Omega^* + U^*$ $\qquad$ ↳ supermodularity!

Aside $\quad V \neq U + \Omega$

$$V = \max_{\substack{c,s \\ \text{s.t. } c+s=k}} U + \Omega$$ but $V^* = U^* + \Omega^*$ b/c same $-\lambda$.

p reacts?

Final Q =

$V_{p\lambda}^* = -V_p(k,p)$

$dV_p^* = -V_{pk}\,dk - V_{pp}\,dp$

$$= -V_{pk}\left[\frac{1}{V_{kk}}\,d\lambda - \frac{V_{pk}}{V_{kk}}\,dp\right] - V_{pp}\,dp.$$

$$= -\frac{V_{pk}}{V_{kk}}\,d\lambda + \left[\frac{(V_{pk})^2}{V_{kk}} - V_{pp}\right]dp$$

PTO →

$dV_p^*(\lambda, p) = V_{p\lambda}^* \, d\lambda + V_{pp}^* \, dp$

Vertical addition - easy?
$V_{pp} \ge 0$, → get condition on V_{pp}^*
Use horizontal addition

$\Rightarrow V_{p\lambda}^* = -\frac{V_{pk}}{V_{kk}}$

$$V_{pp}^* = \frac{V_{pk}^2 - V_{kk}V_{pp}}{V_{kk}}$$

<u>Concavity of the consumption f^n</u>

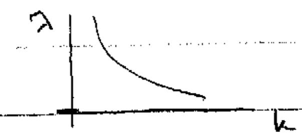
$c = u_\lambda^*(\lambda), \quad k = V_\lambda^*(\lambda)$

$dc = u_{\lambda\lambda}^* \, d\lambda$

$dk = V_{\lambda\lambda}^* \, d\lambda$

<u>MPC out of wealth</u> $\left\{ \frac{dc}{dk} = \frac{u_{\lambda\lambda}^*(\lambda)}{V_{\lambda\lambda}^*(\lambda)} \right.$

$\frac{d^2c}{dk^2} \le 0$



∴ Need $\frac{d}{d\lambda}\left(\frac{dc}{dk}\right) \ge 0$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{u_{\lambda\lambda}^*}{V_{\lambda\lambda}^*}\right) = \frac{u_{\lambda\lambda\lambda}^* V_{\lambda\lambda}^* - V_{\lambda\lambda\lambda}^* u_{\lambda\lambda}^*}{(V_{\lambda\lambda}^*)^2} \overset{?}{\ge} 0.$$

$\times \left(\frac{\lambda V_{\lambda\lambda}^*}{u_{\lambda\lambda}^*}\right) \ge 0$

$\Rightarrow \frac{\lambda V_{\lambda\lambda\lambda}^*}{V_{\lambda\lambda}^*} \ge -\frac{\lambda u_{\lambda\lambda\lambda}^*}{u_{\lambda\lambda}^*}$ <u>Prudence</u> in V greater than in u.

Conjugate to
Strategy: need $-\frac{\lambda u_{\lambda\lambda\lambda}^*}{u_{\lambda\lambda}^*} = a$. (1) (HARA)

$-\frac{\lambda V_{\lambda\lambda\lambda}}{V_{\lambda\lambda}^*} \ge a$. (2)

(2) $-V_{kk}\left(\frac{-V_{kkk}}{V_{kk}^3}\right)V_{kk} \geq \theta$

$$\frac{V_{kkk}V_k}{V_{kk}^2} \geq \theta$$

Horizontally: cde treatment

$\lambda V^{\circ}_{\lambda\lambda\lambda} \geq -\theta V^{\circ}_{\lambda\lambda}$

$\lambda u^{\circ}_{\lambda\lambda\lambda} = -\theta u^{\circ}_{\lambda\lambda}$

$\lambda \Omega^{\circ}_{\lambda\lambda\lambda} \geq -\theta \Omega^{\circ}_{\lambda\lambda}$

$\lambda V^{\circ}_{\lambda\lambda\lambda} \geq -\theta V^{\circ}_{\lambda\lambda}$. b/c $V^{\circ} = u^{\circ} + \Omega^{\circ}$

$\begin{bmatrix} V_{kkk} & \sqrt{\theta} V_{kk} \\ \sqrt{\theta} V_{kk} & V_k \end{bmatrix}$ → p.d of this matrix

↳ adds nicely

↳ fails for $\theta < 0$